University “Alexandru Ioan Cuza” from Iaşi

Faculty of Physics

GAUGE-AXIOMATICS IN THE GEOMETRO-DYNAMICS OF PHYSICAL FIELDS

Thesis Summary

Scientific coordinator
Prof. Univ. Dr. Ciprian Dariescu

Candidate
Drd. Ciprian Crețu

Iași –2015

To whom it may concern

....

"ALEXANDRU IOAN CUZA" UNIVERSITY OF IAȘI
FACULTY OF PHYSICS

This is to let you know that in the day of …..…….. 2015, at ………. o'clock, room…………, the PhD candidate **Ciprian Crețu** will present in public hearing the PhD thesis:

„ **GAUGE-AXIOMATICS IN THE GEOMETRO-DYNAMICS OF PHYSICAL FIELDS**"

in order to obtain the scientific title of PhD in the fundamental field of Exact Sciences, domain of Physics.

The PhD committee will have the following members:

President:

Prof. univ. dr. Diana MARDARE, Manager of PhD School, Faculty of Physics, "Alexandru Ioan Cuza" University of Iasi

Scientific coordinator:

Prof. univ. dr. Ciprian DARIESCU, Faculty of Physics, "Alexandru Ioan Cuza" University of Iasi

Referents:

Prof. univ. dr. Dumitru VULCANOV, Faculty of Physics, West University from Timișoara

Prof. univ. dr. Irina RADINSCHI, Department of Physics, Faculty of Machines Constructions and Industrial Management, "Gheorghe Asachi" Technical University from Iasi

Conf. univ. dr. Mircea-Claudiu CRÂȘMĂREANU, Faculty of Mathematics, "Alexandru Ioan Cuza" University of Iasi

Content of the thesis

Introduction…………………………………………………………… 5

Bibliography Introduction…………………………………………….. 10

I. Fundamental mathematical elements in the geometrisation and field theories

I.1. Elements of differential geometry …………………………………… 13

I.2. The isometry group for a class of planary symmetric metrics in null-coordinate formulation ………………………………………. 29

Conclusions Chapter I………………………………………………………. 41

Bibliography Chapter I……………………………………………………….. 41

II. Beil metrics, geodesics and the connection with the electrodynamics

II.1. Finsler spaces, Lagrange spaces……………………………………….. 43

II.2. Electrodynamics from modified Schwarzschild metric …………………. 50

Conclusions Chapter II……………………………………………………… 56

Bibliography Chapter II……………………………………………………… 57

III. Invariance geometrisation of the gauging of internal symmetries

III. 1. Internal gauging groups ……………………………………….……. 59

III. 2. The Lorentz-invariant U(1)-gauge theory of scalars in static external fields and thermal properties…………………………………..…. 71

III. 3. Analytic study of fermions in graphene; Heun functions and beyond … 81

Conclusions Chapter III……………………………………………………… 91

Bibliography Chapter III……………………………………………………… 92

IV. External symmetries in locally covariant formulation with applications in extra-dimensions and Schrödinger cosmology

IV.1. Gauge invariance geometry ……………………………………….…. 95

IV.2.	Geometrisation principles in extra dimensions ………………………..…	104
IV.3.	The quantum treatment of the 5D-warped Friedmann–Robertson–Walker Universe in Schrödinger picture …………………………………	110
IV.4.	On a Schrödinger-like equation with some special potential ……………	120
Conclusions Chapter IV………………………………………………………..		127
Bibliography Chapter IV…………………………………………………..…		127
General conclusions……………………………………………………………….		130
Bibliography (in alphabetical order) ………………………………………………		133
List of personal publications ……………………………………………………		139
Papers presented at international and national conferences ………………………		140

Introduction

The PhD thesis entitled „Gauging Axiomatics in the geometrical-dynamical theories of the main physical fields" is structured in four chapters. Each chapter can be independently analyzed from the others, ending with a conclusions section and a bibliographic list.

In the first chapter, entitled „Fundamental Mathematical elements in the geometrisation and field theories", I briefly presented notions and results of differential geometry [1], grouped in section I.1. „Elements of differential geometry". The Killing vectors are associated with the space-time symmetries, these always being of a fundamental importance in Physics. In particular, the planar symmetry originates in some exact solutions of Einstein's equations [2]. The metrics with planar symmetry can generalize the well-known Robertson-Walker geometries [3] for spaces with extra-dimensions. "The isometry group for a class of planary symmetric metrics in null-coordinate formulation " is the title of section I.2. and the subject of a paper presented at the international conference TIM14 "Physics without frontiers".

In chapter II, entitled „Beil metrics, geodesics and the connection with the electrodynamics", I analyzed the implications of this type of metrics in the theories of the unification of the physical fields. The models built on the basis of these metrics are the Finsler spaces. The theory of gauge transformations in Finsler spaces is applied in the general relativity [4]. These transformations produce new metrics which correspond to the geometrodynamics introduction of additional physical fields. The equation of the geodesic in the transformed space is equivalent to the motion equation [5] in the initial space where the additional field is included by a so called term of force. An example is given by a special transformation and by the resulting metric in which, the electromagnetic potential is more likely connected to the parameters of gauge transformation than to the traditional gauge potential. This actually means that the electromagnetic field corresponds to an additional connection on the basic variety and not only to a simple curving term [6]. The theoretical elements necessary to the study [7] already mentioned are grouped in section II.1, entitled „Finsler spaces, Lagrange spaces". "Electrodynamics from modified Schwarzschild metric" is the title of section II.2. and the subject of a paper presented at the international conference TIM13 „Physics without frontiers".

The symmetry properties of the elementary particles are studied with the help of the external or spatial symmetry groups (Lorenz or Poincare groups) and with the help of internal symmetry groups (the unitary groups U(1), SU(3), SU(n). We keep taking into consideration transformations that do not change space-time coordinates but which change the field functions. Such transformations refer to the internal properties of the fields and particles for this reason being called internal transformations [8]. These are the very subject for study in chapter III, entitled „ Invariance geometrisation of the gauging of internal symmetries"

The invariance of a langrangian for a group of global transformation is not automatically kept for the local transformations of the same group. In order to find a lagrangian to satisfy both conditions new types of fields are introduced , the so called gauge fields which modify the lagrangian in such a way as to also become invariant to local transformations (gauge) [8], situation presented in section III.1. „Internal gauging groups". The subject of a paper, published in Buletinul Institutului Politehnic from Iasi (2012), is presented in section III.2. and the paper is entitled „ The Lorentz-invariant U(1)-gauge theory of scalars in static external fields and thermal properties". The section III.3. with the title „ Analytic study of fermions in graphene; Heun functions and beyond" presents the subject of a paper published in Romanian Journal of Physics"(2013).

In chapter IV, entitled „External symmetries in locally covariant formulation with applications in extra-dimensions and Schrödinger cosmology", in the first part we apply the mathematical notions introduced so far in the formulation of the gauge theories. First, in the abelian case, of the Maxwell equations, then in the case of the invariant gauge geometry [9]. In the section IV.1., entitled „ Gauge invariance geometry", we briefly present the necessary theories. After the Randall-Sundrum model had appeared, different braneworld scenarios were formulated starting from the idea that our Universe, in which the particles of the standard model are caught, is incorporated in a hyperspace (bulk) of a bigger dimension [10]. Since in the RS model the matter is practically excluded from the membrane, limitations mechanisms were proposed among which the most familiar is the coupling to a scalar field. Aspects of this problematic are approached in the second part of the chapter IV, in section IV.2., entitled „ Geometrisation principles in extra dimensions"[10]. The subject of a paper, published in the International Journal of Theoretical Physics (2012), is presented in section IV.3, entitled „ The quantum treatment of the 5D-warped Friedmann–Robertson–Walker Universe in Schrödinger picture". At the same time, section IV.4., entitled „ On a Schrödinger-like equation with some special potential", represents the content of a paper published in Buletinul Institutului Politehnic from Iasi, (2013).

Selective bibliography Introduction

[1] S. Kobayashi, Nomizu K., *Foundation of Differential Geometry*, Interscience Publ., (1969).

[2] D. Kramer, H. Stephani, E. Herlt, *Exact solutions of Einstein's field equations*, Berlin, Cambridge University Press, (1980).

[3] H.P. Robertson, *Relativity and Cosmology*, W.B. Saunders, London, (1968).

[4] S. Ikeda, *Adavanced Studies in Applied Geometry*, Seizansha, Sagamihara, Japan, (1995).

[5] D.J. Griffiths, *Introduction to Electrodynamics*, Prentice Hall, New Jersey, (1999).

[6] R.G. Beil, *Electrodynamics from a metric*, Int. J.of Theor. Phys., Vol. 26, Issue 2, p. 189, (1987).

[7] R. Miron, M. Anastasiei, *The geometry of Lagrange spaces:Theory and Applications*, Kluwer Acad. Publ., FTPH, no.59, (1994).

[8] Gh. Zet, *Simetrii unitare și teorii gauge*, Ed. Gh. Asachi, Iași, (1998).

[9] M. Göckeler, T. Schücker, *Differential geometry, gauge theories, and gravity*, Cambridge, Cambridge University Press, (1987).

[10] L. Randall, R. Sundrum, *Large Mass Hierarchy from a Small Extra Dimension*, Phys. Rev. Lett., Vol. 83, Issue 17, p. 3370, (1999).

Chapter I

Fundamental mathematical elements in the geometrization and field theories

I.1. Elements of Differential Geometry

I.2. The isometry group for a class of planary symmetric metrics in null-coordinate formulation

The origins of the planar symmetries are to be found in some exact solutions of Einstein's equations for astrophysical or cosmological objects relatively modern [1] such as: flat walls, empty cylinders or cords and different aspects of the critical collapse of non-spherical symmetry. In addition, such metrics can generalize the well-known Robertson-Walker geometries for spaces with extra-dimensions [2]. The study of the symmetrical physical fields reveal the simplicity and the repetition of some phenomena. In the case of an extremely symmetric system we find equations relatively easy to solve with solutions which have special properties. In a Riemannian variety we have symmetry if the movement of the points to a certain direction does not change the distances between them. Imposing the invariant of the metric for the movements of infinitesimal translation [3]

$$\bar{x}^i = x^i + \delta x^i ,$$

in the direction of the vector field X^i

$$\delta x^i = X^i d\lambda ,$$

we have the following relation

$$\delta(ds^2) = \delta\left(g_{ik} dx^i dx^k\right) = 0.$$

After calculations, we obtain symmetry if the system of differential equations

$$g_{ik,l} X^l + g_{kl} X^l_{,i} + g_{il} X^l_{,k} = 0$$

has a solution. Calculating the Lie derivative of the metric g, along the vector field $X = X^l \partial_l$, we get

$$L_X g = L_X\left(g_{ik} dx^i dx^k\right) = (g_{ik,l} X^l + g_{lk} X^l_{,i} + g_{il} X^l_{,k}) dx^i dx^k.$$

Then X is called Killing vector field relative to g if

$$L_X g = 0,$$

where $L_X(.)$is the Lie derivative, which means

$$(L_X g)_{ik} = g_{ik,l} X^l + g_{kl} X^l_{,i} + g_{il} X^l_{,k} , \text{ since } g_{kl} = g_{lk}. \qquad (1)$$

We consider the Riemannian variety M, endowed with planary symmetric metric

$$ds^2 = e^{2f(z,t)}(dx^2 + dy^2) + dz^2 - dt^2. \qquad (2)$$

Performing the null-type substitutions [4]

$$u = \frac{t - z}{\sqrt{2}}, \ v = \frac{t + z}{\sqrt{2}},$$

metric it can be written as

$$ds^2 = g_{ik} dx^i dx^k = e^{2f(u,v)} \delta_{AB} dx^A dx^B - 2du dv, \qquad (3)$$

where $u = x^3, \ v = x^4, \ A, B = \overline{1,2}$.

In order to work out the Killing vector field we start with the Lie derivative (1) imposing the isometry condition [5]

$$g_{ik,l}X^l + g_{kl}X^l_{,i} + g_{il}X^l_{,k} = 0,\ i,k = \overline{1,4}\ . \tag{4}$$

Finally, for the analyzed metric we have derived the following Killing generators $\widehat{K}_1, \ldots, \widehat{K}_6$, under the form

$$\widehat{K}_1 = \partial_x, \qquad Z_0^1, \qquad \widehat{K}_2 = \partial_y, \qquad Z_0^2, \qquad \widehat{K}_3 = y\partial_x - x\partial_y, \qquad \omega,$$
$$\widehat{K}_4 = \partial_v, \qquad A_0^4, \qquad \widehat{K}_5 = U\partial_x + x\partial_v, \qquad \widehat{K}_6 = U\partial_y + y\partial_v.$$

These correspond to the general Killing vectorial field [5]

$$\vec{X} = Z_0^1\partial_x + \omega y\partial_x + A_1^4 U\partial_x + Z_0^2\partial_y - \omega x\partial_y + A_2^4 V\partial_y + A_1^4 x\partial_v + A_2^4 y\partial_v + A_0^4\partial_v$$

and to the respective infinitesimal transformations

$$\frac{dx}{d\mu} = U, \qquad \frac{dv}{d\mu} = x, \qquad \frac{dy}{d\mu} = V, \qquad \frac{dv}{d\mu} = y,$$
$$\frac{dx}{dv} = \frac{U}{x}\ , \qquad U \sim x^2\ , \qquad \frac{dy}{dv} = \frac{V}{y}\ , \qquad V \sim y^2\ .$$

Conclusions Chapter I

The origins of planar symmetry reside in some exact solutions of Einstein equations for relatively modern astrophysical or cosmologic objects such as: planar walls, hollow cylinders or strings and various aspects of non-spherically symmetric critical collapse. In addition, such metrics can generalize the well-known Robertson-Walker geometries to spaces with extra-dimensions and more intriguing causalities.

Starting from a class of planary symmetric metrics in null-coordinate formulation I obtained the Killing vectors. By using an adequate change of coordinates, I calculated the Lie derivative of the metric

$$ds^2 = e^{2f(z,t)}(dx^2 + dy^2) + dz^2 - dt^2\ .$$

By respecting the conditions of integrability I solved the corresponding Killing equations.

These results are the very subject of the paper [6].

Selective bibliography Chapter I

[1] D. Kramer, H. Stephani, E. Herlt, *Exact solutions of Einstein's field equations*, Berlin, Cambridge University Press, (1980).

[2] H.P. Robertson, *Relativity and Cosmology*, W.B. Saunders, London, (1968).

[3] H. Stephani, *General Relativity. An introduction to the theory of the gravitational field*, Cambridge University Press, Cambridge, (1982).

[4] C. Dariescu, *Planary Symmetric Static Worlds with Massless Scalar Source*, Foundations of Physics, Vol. 26, No. 8, p. 1069, (1996).

[5] I.Aștefănoaei, C.Dariescu, M.A.Dariescu, *Modele speciale de Univers și patologii spațio-temporale*, Ed.Univ. "Al.I.Cuza" Iasi (2007).
[6] C. Crețu, C. Dariescu, *On the Isometry Group for a Class of Planary Symmetric Metrics in Null-coordinate Formulation* , TIM14 Physics Conference, Univ. de Vest, Timișoara, (2014).

Chapter II

Beil metrics, geodesics and the connection with the electrodynamics

II. 1. Finsler spaces, Lagrange spaces

II.2. Electrodynamics from modified Schwarzschild metric

Let be a Minkowski space in which a charged particle is moving in the presence of an electromagnetic field. The motion is described by Lorentz equations [2]. R.G. Beil proposes a modification of the metric of the space such that the Lorentz equations become the equations of geodesics in the new metric. [3]. The electromagnetic force then results from the new connection. I resume the main ideas and results from the article [3] of R.G. Beil, using his notations. He considered a Minkowski space [1]

$$c^2 d\tau^2 = \eta_{\mu\nu} dx^\mu dx^\nu \,, \tag{1}$$

where the metric signature is (+1,-1,-1,-1). The trajectory parameter is proper time τ, the position of the particle is given by $x^\mu(\tau)$, the speed and acceleration are $v^\mu = dx^\mu/d\tau$ and $a^\mu = dv^\mu/d\tau$. The particle is driven by an electromagnetic field from a potential $A_\mu(x)$ the equation of motion (Lorentz) [2] is

$$a^\mu = e(mc)^{-1}\eta^{\mu\nu}F_{\nu\lambda}v^\lambda \tag{2}$$

where $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ is the electromagnetic tensor.
The trajectory described by this equation is not a geodesic in Minkowski space. Regarding the particle, the new metric produces a change of scale along the trajectory, which is characterized by a new trajectory parameter $\bar{\tau}$, so that

$$c^2 d\bar{\tau}^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu \,. \tag{3}$$

In the new space, the speed and acceleration are $\bar{v}^\mu = dx^\mu/d\bar{\tau} = (dx^\mu/d\tau)b$ and $\bar{a}^\mu = d\bar{v}^\mu/d\bar{\tau} = b^2 a^\mu + v^\mu(db/d\bar{\tau})$. The scale function is considered $b = d\tau/d\bar{\tau}$.

The main idea is the assumption that the form of the metric $\bar{g}_{\mu\nu}$ is

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu} + kB_\mu B_\nu \,, \tag{4}$$

where k is a constant to be determined and vector B_μ is related to the electromagnetic potential in Minkowski space, A_μ. By replacing (4) in (3), we get

$$c^2 d\bar{\tau}^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = \bar{g}_{\mu\nu} v^\mu v^\nu d\tau^2 = \left[c^2 + k\left(B_\mu v^\mu\right)^2\right] d\tau^2,$$

from which

$$b = \left[1 + kc^{-2}\left(B_\mu v^\mu\right)^2\right]^{-1/2}. \tag{5}$$

We notice, from relation (5), that b depends on the point and speed. The geodesic equation in the new metric is [3] $\bar{a}^\mu + \bar{\Gamma}^\mu_{\alpha\beta}\bar{v}^\alpha\bar{v}^\beta = 0$ or, after the computation of Christoffel connection,

$$a^{\mu} + kb^2\left[B^{\mu} - \frac{v^{\mu}}{c^2}(B_{\alpha}v^{\alpha})\right]d(B_{\alpha}v^{\alpha})/d\tau + k\eta^{\mu\lambda}H_{\beta\lambda}(B_{\alpha}v^{\alpha})v^{\beta} = 0, \qquad (6)$$

where $H_{\mu\nu} = B_{\nu,\mu} - B_{\mu,\nu}$. By comparing equation (6) with Lorentz equation (2) we get their coincidence, only if the vector B_{μ} is connected to the electromagnetic potential A_{μ} through a gauge transformation under the form of

$$B_{\mu} = A_{\mu} + \frac{\partial\Lambda}{\partial x^{\mu}} \qquad (7)$$

and we have the relation

$$k(B_{\alpha}v^{\alpha}) = -e(mc)^{-1}. \qquad (8)$$

In these conditions we also get $H_{\mu\nu} = F_{\mu,\nu}$. The identification of k follows after the study of the field equations. We apply the R. G. Beil method [3] to the situation in which the Minkowsky metric $\eta_{\mu\nu}$ is replaced by the Schwarzschild metric.

The case of Schwarzschild metric

We consider, instead of a flat metric $\eta_{\mu\nu}$, a Schwarzschild metric $g_{\alpha\beta}$. The expression for Schwarzschild metric with static spatial symmetry is [4]

$$ds^2 = \left(1 - \frac{r_g}{r}\right)(dx^0)^2 - \left(1 - \frac{r_g}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + sin^2\theta d\varphi^2), \qquad (9)$$

where r_g is the gravitational radius and $x^0 = ct$.

The Lorentz equation of the particle motion in $g_{\alpha\beta}$ space with potential A_{μ} is

$$a^{\mu} = e(mc)^{-1}g^{\mu\nu}F_{\nu\lambda}v^{\lambda} \qquad (5)$$

where $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$. We transform $\bar{g}_{\alpha\beta} = g_{\alpha\beta} + kB_{\alpha}B_{\beta}$ into spherical coordinates

$$\bar{g}_{\alpha\beta} = \begin{pmatrix} 1 - r_g/r + kB_0^2 & kB_0B_1 & kB_0B_2 & kB_0B_3 \\ kB_1B_0 & -(1 - r_g/r)^{-1} + kB_1^2 & kB_1B_2 & kB_1B_3 \\ kB_2B_0 & kB_2B_1 & -r^2 + kB_2^2 & kB_2B_3 \\ kB_3B_0 & kB_3B_1 & kB_3B_2 & -r^2sin^2\theta + kB_3^2 \end{pmatrix}.$$

Considering $B = (B_0(r), 0,0,0)$, then

$$\bar{g}_{\alpha\beta} = \begin{pmatrix} 1 - r_g/r + kB_0^2(r) & 0 & 0 & 0 \\ 0 & -(1 - r_g/r)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2sin^2\theta \end{pmatrix}$$

By using the technique from [3] it is obtained the following equation of the geodesic

$$a^{\mu} + kb^2\left[B^{\mu} - \frac{v^{\mu}}{c^2}(B_{\alpha}v^{\alpha})\right]d(B_{\alpha}v^{\alpha})/d\tau + kg^{\mu\lambda}H_{\beta\lambda}(B_{\alpha}v^{\alpha})v^{\beta} = 0 \qquad (6)$$

where $H_{\mu\nu} = B_{\nu,\mu} - B_{\mu,\nu}$. It takes place the coincidence of the geodesic equation with the Lorentz equation only if the B_{μ} is connected to the electromagnetic potential electromagnetic A_{μ} through a gauge transformation under the form

$$B_{\mu} = A_{\mu} + \frac{\partial\Lambda}{\partial x^{\mu}} \qquad (7)$$

and it takes place the relation

$$k(B_{\alpha}v^{\alpha}) = -e(mc)^{-1}. \qquad (8)$$

The dependence on the point (x) and speed (v) of the scale factor b is transmitted to vector B_μ and even further to the metric $\bar{g}_{\mu\nu}$. Being dependent on point and speed, $\bar{g}_{\mu\nu}(x,v)$ is a generalized Lagrange metric [5].

The metric $\bar{g}_{\mu\nu}$ is reduced to a Finsler metric if it is 0-omogenous in λ, condition reduces to equality $B_\mu(x,\lambda v) = B_\mu(x,v)$. In this case we obtain Finsler function of the form

$$F(x^\mu, v^\mu) = (\bar{g}_{\mu\nu} v^\mu v^\nu)^{1/2} .$$

Further, the metric $\bar{g}_{\mu\nu}$ may be reduced to a Riemannian metric if B_μ depends only on the point x. Identification of k follows the study of field equations [2].

The non-zero components of the Ricci tensor for the metric $\bar{g}_{\alpha\beta}$ are:

$$\begin{aligned} R_{11} = & -k\left[4r^3\left(r - r_g + krB_0^2\right)\right]^{-1}\left(-8r^3 B_0 B_0'' r_g + 4r^2 B_0 B_0'' r_g^2 - 4r^3 k B_0^3 B_0'' r_g - 18r^2 r_g B_0 B_0'\right) \\ & -k\left[4r^3\left(r - r_g + krB_0^2\right)\right]^{-1}\left(10 r r_g^2 B_0 B_0' - 6r^2 r_g k B_0^3 B_0' + 4r^4 B_0'^2 + 8r^3 B_0 B_0' + r_g^2 B_0^2\right) \\ & -k\left[4r^3\left(r - r_g + krB_0^2\right)\right]^{-1}\left(8kr^3 B_0^3 B_0' - 8r^3 B_0'^2 r_g + 4r^2 r_g^2 B_0'^2 + 4r^2 B_0' B_0'' + 4kr^4 B_0^3 B_0''\right), \end{aligned}$$

$$\begin{aligned} R_{22} = & \left[4r\left(r - r_g + krB_0^2\right)^2\left(r - r_g\right)\right]^{-1}\left(-8kr^3 B_0 B_0'' r_g + 4r^2 B_0 B_0'' r_g^2 - 4r^3 k B_0^3 B_0'' r_g - 2r^2 r_g B_0 B_0'\right) + \\ & + \left[4r\left(r - r_g + krB_0^2\right)^2\left(r - r_g\right)\right]^{-1}\left(2 r r_g^2 B_0 B_0' + 2r^2 r_g k B_0^3 B_0' + 4r^4 B_0'^2 - 8r^3 r_g B_0'^2 + 4r^4 B_0 B_0''\right) + \\ & + \left[4r\left(r - r_g + krB_0^2\right)^2\left(r - r_g\right)\right]^{-1}\left(4r^4 B_0'^2 + 4kr^4 B_0^3 B_0'' + 4 r r_g B_0^2 - 3r_g^2 B_0^2 + 4kr r_g B_0^4\right), \end{aligned}$$

$$R_{33} = \left[2\left(r - r_g + krB_0^2\right)\right]^{-1}\left(2r^2 B_0' - 2r r_g B_0' - r_g B_0\right) k B_0,$$

$$R_{44} = \left[2\left(r - r_g + krB_0^2\right)\right]^{-1} sin^2\theta \, k B_0 \left(2r^2 B_0' - 2r r_g B_0' - r_g B_0\right),$$

where $B_0' = dB_0/dr$ and $B_0'' = d^2 B_0/dr^2$.

The curvature scalar $R = \bar{g}^{\nu\mu} R_{\nu\mu}$ is

$$\begin{aligned} R = & -k\left[2r^2\left(r - r_g + krB_0^2\right)^2\right]^{-1}\left(-8r^3 B_0 B_0'' r_g + 4r^2 B_0 B_0'' r_g^2 - 4r^3 k B_0^3 B_0'' r_g - 18r^2 r_g B_0 B_0'\right) \\ & -k\left[2r^2\left(r - r_g + krB_0^2\right)^2\right]^{-1}\left(10 r r_g^2 B_0 B_0' - 6r^2 r_g k B_0^3 B_0' + 4r^4 B_0'^2 + 8r^3 B_0 B_0' + r_g^2 B_0^2\right) - \\ & -k\left[2r^2\left(r - r_g + krB_0^2\right)^2\right]^{-1}\left(8kr^3 B_0^3 B_0' - 8r^3 B_0'^2 r_g + 4r^2 r_g^2 B_0'^2 + 4r^2 B_0' B_0'' + 4kr^4 B_0^3 B_0''\right). \end{aligned}$$

The non-zero components of the Einstein tensor, $G_{\nu\mu} = R_{\nu\mu} - \frac{1}{2}\bar{g}_{\nu\mu} R$ are:

$$G_{22} = -kB_0\left[r\left(r - r_g\right)\left(r - r_g + krB_0^2\right)\right]^{-1}\left(2r^2 B_0' - 2r_g B_0' r - r_g B_0\right),$$

$$G_{33} = -k\left[4\left(r - r_g + krB_0^2\right)^2\right]^{-1}\left(4B_0B_0'r^3 - 10r_gB_0B_0'r^2 + 4kB_0^3B_0'r^3 + 6rr_g^2B_0B_0'\right)$$
$$-k\left[4\left(r - r_g + krB_0^2\right)^2\right]^{-1}\left(-2kr_gB_0^3B_0'r^2 + 2rr_gB_0^2 - r_g^2B_0^2 + 2kr_gB_0^4 - 8r^3r_gB_0B_0''\right)$$
$$-k\left[4\left(r - r_g + krB_0^2\right)^2\right]^{-1}\left(4r^2B_0B_0''r_g^2 - 4kr^3r_gB_0^3B_0'' + 4r^4B_0'' - 8r^3r_gB_0'^2 + 4r^2r_g^2B_0'^2\right)$$
$$-k\left[4\left(r - r_g + krB_0^2\right)^2\right]^{-1}\left(4r^4B_0B_0'' + 4kr^4B_0^3B_0''\right),$$

$$G_{44} = -k\,sin^2\theta\left[4\left(r - r_g + krB_0^2\right)^2\right]^{-1}\left(4B_0B_0'r^3 - 10r_gB_0B_0'r^2 + 4kB_0^3B_0'r^3 + 6rr_g^2B_0B_0'\right)$$
$$-k\,sin^2\theta\left[4\left(r - r_g + krB_0^2\right)^2\right]^{-1}\left(-2kr_gB_0^3B_0'r^2 + 2rr_gB_0^2 - 2rr_gB_0^2 - r_g^2B_0^2 + 2krr_gB_0^4\right)$$
$$-k\,sin^2\theta\left[4\left(r - r_g + krB_0^2\right)^2\right]^{-1}\left(4r^2B_0B_0''r_g^2 - 4kr^3r_gB_0^3B_0'' + 4r^4B_0'^2 - 8r^3r_gB_0'^2\right)$$
$$-k\,sin^2\theta\left[4\left(r - r_g + krB_0^2\right)^2\right]^{-1}\left(4r^2r_g^2B_0'^2 + 4r^4B_0B_0'' + 4kr^4B_0^3B_0''\right).$$

The field equations for a particle in an electromagnetic field of potential $A_\mu(x)$ are [5]

$$G_{\eta\gamma} = 8\pi\kappa c^{-4}\left(\rho_0\bar{v}_\eta\bar{v}_\gamma + \bar{T}_{\eta\gamma}\right),$$

with κ the gravitational constant and ρ_0 the proper matter density.

According to [3], for $k = 4\kappa c^{-4}$, the electromagnetic energy tensor $\bar{T}_{\eta\gamma}$is a part of Einstein tensor, i.e. metric comes from $\bar{g}_{\alpha\beta}$.

Conclusions Chapter II

In section II.2., using an idea of R.G. Beil, we modify the Minkowski metric of the space in which a charged particle moves according to Lorentz equation. The modification is such that, with the new metric and in the new space, the particle moves on a geodesic. The process of obtaining the metric appears like a gauge transformation. The dependence on the point (*x*) and speed (*v*) of the scale factor b is transmitted to vector B_μ and even further to the metric $\bar{g}_{\mu\nu}$. Being dependent on point and speed, $\bar{g}_{\mu\nu}(x, v)$ is a generalized Lagrange metric.

The dependence on the point (*x*) and speed (*v*) of the scale factor b is transmitted to vector B_μ and even further to the metric $\bar{g}_{\mu\nu}$. Being dependent on point and speed, $\bar{g}_{\mu\nu}(x, v)$ is a generalized Lagrange metric [5].

The metric $\bar{g}_{\mu\nu}$ is reduced to a Finsler metric if it is 0-omogenous in λ, condition reduces to equality $B_\mu(x, \lambda v) = B_\mu(x, v)$. In this case we obtain Finsler function of the form

$$F(x^\mu, v^\mu) = (\bar{g}_{\mu\nu}v^\mu v^\nu)^{1/2}.$$

Further, the metric $\bar{g}_{\mu\nu}$ may be reduced to a Riemannian metric if B_μ depends only on the point x. Electromagnetic energy tensor $\bar{T}_{\eta\gamma}$ is part of the Einstein tensor, so comes from metric $\bar{g}_{\alpha\beta}$.

These results are the very subject of the paper [6].

Selective bibliography Chapter II

[1] S. Ikeda, *Adavanced Studies in Applied Geometry*, Seizansha, Sagamihara, Japan, (1995).
[2] A.O. Barut, *Electrodynamics and classical theory of fields and particles*, Dover Publications, New York, (1980).
[3] R. G. Beil, *Electrodynamics from a metric*, Int. J.of Theor. Phys.,Vol. 26, Issue 2, p. 189, (1987).
[4] Gh. Munteanu, V. Balan, *Lectii de teoria relativitatii*, Editura Brena, Bucuresti (2000).
[5] R. Miron, M. Anastasiei, *The geometry of Lagrange spaces:Theory and Applications*, Kluwer Acad. Publ., FTPH, no. 59, (1994).
[6] C. Crețu, *Electrodynamics from modified Schwarzschild metric* , TIM13 Physics Conference, Univ. de Vest, Timișoara, (2013).

Chapter III

Invariance geometrisation of the gauging of internal symmetries

III. 1. Internal gauging groups

III.2. The Lorentz-invariant U(1)-gauge theory of scalars in static external fields and thermal properties

The discovery of the quantum Hall effects, both integer and fractional [1], opened a new area of investigations, in the physics of two-dimensional evolution of bosons and fermions. In the case of the integer quantum Hall effect (IQHE), the quantized conductivity appears to be the integer multiplicity of q^2/h; a simple combination of fundamental constants. This effect does not depend on specific parameters of the material, being connected with the Landau energy levels, characterizing the electrons evolving in magnetic fields. It is important to add, however, that impurities and boundary conditions, which remove the Landau levels degeneracy, play a vital role in the quantum Hall effect [1]. On the other hand, the so-called fractional quantum Hall effect (FQHE) is an example of the new physics that has emerged in recent years as a result of active research in quantum-confined carriers in semiconductor hetero-structures [2].

It was first observed in a high-mobility, two-dimensional, modulation-doped GaAs-(AlGa)As samples prepared by molecular-beam epitaxy [3].

The presence of a magnetic field perpendicular to a two-dimensional electron system quantizes the carriers in-plane motion and transforms their energy spectrum into a set of discrete, highly degenerate levels [4].

With the discovery of the new material called graphene, it has been stated that the charge carries are better described by the relativistic field equations [5].

The aim of the present work is to develop a quantum analysis of relativistic bosons, subjected to a static configuration of orthogonal magnetic and electric fields, at finite temperature.

Klein-Gordon equation

In the usual Cartesian coordinates

$$ds^2 = dx^2 + dy^2 + dz^2 - dt^2,$$

the relativistic complex charged boson of mass m_0, evolving in a static magnetic field orthogonal to a static electric field [6] is described by the well-known $U(1)$- gauge invariant Lagrangian density

$$\boldsymbol{L} = \eta^{\boldsymbol{ij}}(D_i\psi)^* D_j\psi + m_0^2\psi^*\psi\,, \tag{1}$$

where D stands for the $U(1)$- gauge covariant derivative,

$$D_i\psi = \psi_{,i} - \frac{\mathrm{i}q}{\hbar}A_i\psi, \quad D_i\psi^* = \psi^*{}_{,i} + \frac{\mathrm{i}q}{\hbar}A_i\psi^*.$$

We use the convenient gauge,

$$A_x = A_z = 0, \quad A_y = B_0 x, \quad A_4 = \frac{E_0}{c}x,$$

where E_0 and B_0 are the orthogonal electric and magnetic fields.

By employing the usual procedure [6], we come to the corresponding Euler-Lagrange equation

$$\eta^{ij}D_i D_j\psi - \frac{m_0^2c^2}{\hbar^2}\psi = 0, \tag{2}$$

whose explicit form is

$$\eta^{ij}\psi_{,ij} - 2\mathrm{i}\frac{q}{\hbar}B_0 x\psi_{,y} + 2\mathrm{i}\frac{q}{\hbar c^2}E_0 x\psi_{,t}$$
$$- \left[\frac{m_0^2c^2}{\hbar^2} + \frac{q^2x^2}{\hbar^2}\left(B_0^2 - \frac{E_0^2}{c^2}\right)\right]\psi = 0\,, \tag{3}$$

where $\hbar$ and c have been inserted in view of a better comparison of the theoretical predictions with experimental data.

Thus, putting everything together, the wave function gets the explicit form [7]

$$\psi = \left(2^n n!\sqrt{\pi}\right)^{-1/2}\exp\left[-\frac{\rho^2}{2} + \frac{i}{\hbar}(py + p_z z - wt)\right]H_n(\rho) \tag{12}$$

once we impose the condition

$$\left[\frac{\left(wB_0 + pE_0\right)^2}{c^2\beta^2} - p_z^2 - m_0^2c^2\right] = \left(2n+1\right)q\beta\hbar,$$

which leads to the energy-quantization relation

$$w_n = -p\frac{E_0}{B_0} \pm \frac{\beta}{B_0}m_0c^2\sqrt{1 + \frac{p_z^2}{m_0^2c^2} + (2n+1)\frac{q\hbar\beta}{m_0^2c^2}}\;. \tag{13}$$

In the case of an infinite number of energy levels, $N \to \infty$, the same energy spectrum (21), with the notations (22), leads to the classical partition function, namely

$$Z = \sum_{n=0}^{\infty} \exp(-\beta \varepsilon_n) = \frac{\exp\left[-\beta\left(a - \frac{\Omega}{2}\right)\right]}{2\sinh\frac{\beta\,\Omega}{2}} . \tag{30}$$

The characteristic function (24), meaning free energy, becomes

$$F = \frac{p_z^2}{2m_0} - p\frac{E_0}{B_0} + kT\ln\left[2\sinh\frac{\beta\,\Omega}{2}\right] , \tag{31}$$

leading to the following negative magnetization

$$M \equiv -\frac{\partial F}{\partial B_0} = -\left[\mu_{BP}\coth\frac{\beta\Omega}{2} + p\frac{E_0}{B_0^2}\right], \tag{32}$$

which contains, besides the usual *coth*-term multiplying the Bohr-Procopiu magneton [8], an additional Hall-type contribution.

In order to put the main results in a simpler form, we introduce the dimension-less variable

$$x = \frac{\Omega}{2kT} \;. \tag{33}$$

The energy coming from the partition function (30) is

$$E = -\frac{\partial \ln Z}{\partial \beta} = \frac{p_z^2}{2m_0} - p\frac{E_0}{B_0} + kT\,x\coth x, \tag{34}$$

allowing us to compute the heat capacity, represented in the figure 1,

$$C = \frac{\partial E}{\partial T} = k\frac{x^2}{\sinh^2 x}. \tag{35}$$

III. 3. Studiul analitic al fermionilor bazat pe funcții Heun

By “graphene”, one generally denotes one planar layer of carbon atoms, arranged on a honeycomb structure made out of hexagons. Its low-energy excitations are massless, chiral, pseudo-particles, moving with a speed 300 times smaller than the speed of light [8].

As a special feature and also a trademark of Dirac fermion behaviour, which makes graphene a very attractive material from a theoretical point of view, is the anomalous integer quantum Hall effect measured experimentally [9], at room temperature [10]. Because the energetic states of the positrons within the barrier are aligned to the continuous energetic states of the electrons outside the barrier, these carriers are transmitted with unit probability [11]. As a result to the insensitivity to external electrostatic potentials, they evolve in an unusual way in the presence of confining potentials that can be easily produced by disorder.

The properties of chiral massless particles, belonging to the distinct sub-lattices in graphene and described by the Dirac equation near the two points K and K', being an active field of research, in a previous paper , we have considered a strong magnetic induction orthogonal to a weak electrostatic intensity. By employing the perturbation theory, we have derived the first-order transition amplitudes and the corresponding current. Then, we have generalized this analysis for arbitrary static magnetic and electric fields, and concluded that the Dirac-type equation of massless fermions is satisfied by the Heun biconfluent functions. Even though these functions have been intensively worked out in the last years, in situations relevant to physics, chemistry and engineering [12], there are problems when dealing with the general expressions. That is why, for having a better understanding of the physical phenomena, in the present paper, we focus on particularly interesting cases which can be investigated by using the corresponding series expansions, for some ranges of the parameters.

Dirac -type equation and fermions' wave function

In natural units, i.e.$\hbar = c = 1$, the four-dimensional Dirac equation describing a massless fermion evolving in an electric field orthogonal to a magnetic field, oriented along Ox and Oz respectively, is

$$\gamma^i D_i \Psi = 0, \;\; D_i = \partial_i - iqA_i, \tag{1}$$

where the covariant derivatives D_i contains the components of the 4-potential

$$A_2 = B_0 x, \quad A_4 = E_0 x.$$

Putting everything together, we are able to write down the full expression of the wave function (3), up to a normalization factor $\mathcal{N}$,

$$\Psi = \mathcal{N} e^{i(p_y - \omega t)} \exp\left(-\frac{\zeta^2}{2} + a\zeta\right) \times$$

$$\times \begin{pmatrix} \zeta^2 HB_1 \\ \zeta^2 HB_2 \\ -\dfrac{i}{d\lambda^2}\left[2 - (b\lambda^2+1)(\zeta^2 - p\lambda\zeta) + \zeta\dfrac{d}{d\zeta}\right] HB_2 \\ -\dfrac{i}{d\lambda^2}\left[2 + (b\lambda^2-1)(\zeta^2 + p\lambda\zeta) + \zeta\dfrac{d}{d\zeta}\right] HB_1 \end{pmatrix}, \tag{18}$$

where

$$a \equiv (p\lambda)(b\lambda^2) = \sqrt{2(n+2)}\,\frac{b}{d}, \text{ and } HB_{1,2} \equiv HeunB[2, -2a, p^2 d^2 \lambda^6, \mp 2p\lambda; \zeta]. \tag{19}$$

The non-vanishing components of the current density defined as

$$j^i = iq\bar{\Psi}\gamma^i\Psi, \;\text{ cu }\; \bar{\Psi} = \Psi^\dagger \beta, \tag{24}$$

are the electric charge density

$$\rho_e = q\Psi^\dagger\Psi$$

which is generating an electric potential through the Poisson equation and the spatial component, j_y, whose dependence on the external fields intensities is

$$j_y = q\Psi^\dagger \alpha^2 \Psi = \frac{4q}{d}|\mathcal{N}|^2 \frac{\xi^2}{\lambda^2}[p\lambda\zeta - b\lambda^2\zeta^2]\exp(-\zeta^2) =$$

$$= \frac{4}{E_0}|\mathcal{N}|^2 \frac{{x_*}^2}{\lambda^4}\left[\left(p_y + \omega\frac{b}{d}\right)x_* - b{x_*}^2\right]\exp\left(-\frac{{x_*}^2}{\lambda^2}\right) =$$

$$\approx \frac{4q^2}{E_0}|\mathcal{N}|^2\left(x + \frac{\omega}{qE_0}\right)^3 (B_0^2 - E_0^2)(p_y - qB_0 x), \tag{25}$$

ω being quantized as in (17).

Conclusions Chapter III

In section III.2., we deal with the Klein-Gordon equation for bosons evolving in a static configuration of orthogonal magnetic and electric fields. We derive the wave functions and the energy spectrum, similarly to the one reported by Novoselov, in graphene. In the semi-relativistic limit, one can recover the well-known Landau levels. By varying the intensities of the external fields, one may reach the zero energy level, whose existence has a deep influence on the properties of the system of bosons. Finally, at low temperatures, we compute the partition function and the main thermodynamic quantities.

These results are the very subject of the paper [13].

In section III.3., starting with the wave function characterizing massless fermions evolving in orthogonal electric and magnetic fields, written in terms of Heun Biconfluent functions, we analyse some physically interesting cases. When the HeunB function truncates to a polynomial form, one may easily compute the essential components of the conserved current density. For a vanishing electric field, we get the familiar Hermite associated functions and discuss the current dependence on the sample width. In the opposite case, corresponding to an electric static field alone, one has to deal with HeunB functions of complex variable and parameters.

These results are the very subject of the paper [14].

Selective bibliography Chapter III

[1] K. von Klitzing, K. Dorda, M. Pepper, *The Fractional Quantum Hall Effect*, Phys. Rev. Lett.,Vol. 45, p. 494, (1980).

[3] S. Datta, *Electronic Transport in Mesoscopic System*, Cambridge Univ. Press, Cambridge, (1997)

[2] R. B. Laughlin, *Anomalous Quantum Hall effect: An Incompressible Quantum Fluid with Fractionally Charged Excitations*, Phys. Rev. Lett., Vol. 50, p. 1395, (1983).

[5] K. S. Novoselov, E. Mc Cann, S.E. Morozov, *Unconventional Quantum Hall Effect and Berry's Phase of 2 in Bilayer Graphene*, Nature Physics, Vol. 2, p. 177, (2006).

[4] M. A. Dariescu, C. Dariescu, Murariu G., *Topological Quantum Dynamics of Charged Bosons*, Chaos, Solitons and Fractals, Vol. 28, Issue 1, p. 1, (2006).

[6] D.J. Griffiths, *Introduction to Electrodynamics*, Prentice Hall, New Jersey, (1999).

[8] A.O. Barut, *Electrodynamics and classical theory of fields and particles*, Dover Publications, New York, (1980).

[7] L.H. Ryder, *Quantum field theory*, Cambridge University Press, Cambridge, (1985).

[8] A. H. Castro Neto, F. Guinea, N.M.R. Peres, K.S. Novoselov, și A.K. Geim, *The electronic properties of graphene*, Rev Mod. Phys., Vol. **81**, No. 1, p. 109, (2009).

[9] K. S. Novoselov, A. K. Geim, S. V. Morozov, D. Jiang, *Two-dimensional gas of massless Dirac fermions in graphene*, Nature, Vol. **438**, p. 197, (2005).

[10] K. S. Novoselov, și colab., *Room-Temperature Quantum Hall Effect in Graphene*, Science, Vol. 315, No. 5817, p. 1379, (2007).

[11] M. I. Katsnelson, K. S. Novoselov, A. K. Geim, *Chiral tunneling and the Klein paradox in graphene*, Nature Physics, Vol. 2, p. 620, (2006).

[12] F.M. Arscott, *Heun's equation*, in: Ronveaux, A. (ed.) Heun's Differential Equations. Oxford University Press, Oxford (1995).

[13] M.A. Dariescu, O. Buhucianu , C. Creţu, *The Lorentz-invariant U(1)-gauge theory of scalars in static external fields and thermal properties*, Buletinul Institutului Politehnic din Iaşi, Tomul LVIII(LIX), Fasc. 2, p. 43, (2012).
[14] M.A. Dariescu, C. Dariescu, C. Creţu, O. Buhucianu, *Analytic Study of Fermions in Graphene; Heun Functions and Beyond*, Rom.Journ.Phys.,Vol. 58, Nos.7-8, p. 706, Bucharest, (2013).

Chapter IV

External symmetries in locally covariant formulation with applications in extra-dimensions and Schrödinger cosmology

III. 1. Gauge invariance geometry

III.2. Geometrisation principles in extra dimensions

III.3. The quantum treatment of the 5D-warped Friedmann–Robertson– Walker Universe in Schrödinger picture

After the pioneering work of Randall and Sundrum (RS) [1], different braneworld scenarios have been formulated based on the idea that our Universe in which the particles of the standard model are trapped, is embedded in a higher dimensional hyperspace.

Since, in the RS model, the matter is practically expelled from the brane, mechanisms of confinement have been proposed, among which the coupling to a scalar field is the most familiar. While for the four-dimensional Minkowski slices, the warp factor and the scalar wave function have been parametrized in terms of a model superpotential [2], when going to bent branes, one needs to solve in full the Einstein's equations, in order to analyze how bulk modes interact with matter on the domain walls with de Sitter and anti-de Sitter geometries [3].

By assuming that the observed large-scale structure can exist on the brane embedded in an AdS5 bulk supported by the matter energy-momentum tensor of a perfect fluid, in the present work, we discuss the (timeless) Wheeler–De Witt (WDW) equation [4], written for the five dimensional k = 0—FRW Universe, whose warp factor and scale function have been obtained in a previous paper [5].

Warped FRWUniverse

Let us start by considering the 5-dimensional space as being described by the metric

$$ds_5^2 = e^{2F(\tau,w)}\eta_{ik}dx^i dx^k + (dw)^2,\ i,k = \overline{1,4}\ , \tag{1}$$

where $(\eta_{ik}) = \mathrm{diag}[1,1,1,-1]$ is the usual Minkowski metric and the warp factor, e^F, depends on the conformal time τ and on the extra dimension, $x^5 = w$. In order to employ the Cartan formalism, we define the pseudo-orthonormal frame $\{e_a\}_{a=\overline{1,5}}$, whose dual bases is

$$\omega^a = e^F dx^a,\ \omega^5 = dw\ ,$$

so that

$$ds_5^2 = \eta_{ab}\omega^a\omega^b\ ,$$

with $(\eta_{ab}) = \mathrm{diag}[1,1,1,-1,1]$. From the first Cartan equation,

$$d\omega^{\mu} = -F_{|4}\omega^{\mu} \wedge \omega^{4} - F_{|5}\omega^{\mu} \wedge \omega^{5}, \tag{2}$$

$$d\omega^{4} = -F_{|5}\omega^{4} \wedge \omega^{5},$$

$$d\omega^{5} = 0, \tag{3}$$

where $\mu = \overline{1,3}$, pop up the connection coefficients

$$\Gamma_{\alpha 4\alpha} = F_{|4}, \quad \Gamma_{\alpha 5\alpha} = F_{|5}, \quad \Gamma_{454} = -F_{|5}, \tag{4}$$

and, by applying the General Relativity formalism, one comes all the way down to the following Einstein tensor components in the five-dimensional bulk

$$G_{\alpha\alpha} = -\left[2F_{|44} + 3\left(F_{|4}\right)^{2}\right] + 3[F_{|55} + 2\left(F_{|5}\right)^{2}],$$

$$G_{44} = 3\left(F_{|4}\right)^{2} - 3[F_{|55} + 2\left(F_{|5}\right)^{2}], \tag{5}$$

$$G_{55} = -3\left[F_{|44} + 2\left(F_{|4}\right)^{2}\right] + 6\left(F_{|5}\right)^{2}, \qquad G_{45} = -3F_{|54}.$$

As in [5], we are separating the variables in the warp function as

$$F(\tau, w) = f(\tau) + h(w), \tag{6}$$

so that $G_{45} = 0$. One may easily check that a perfect fluid with

$$T_{ab} = (\rho + P)u_a u_b + P\eta_{ab}, \tag{7}$$

in a comoving frame, with $u_4 = -1$ and $(u_\alpha, u_5) = 0$ is a suitable source for the Einstein's equations,

$$G_{ab} + \eta_{ab}\Lambda = \kappa T_{ab}, \tag{8}$$

where Λ and κ stand for the cosmological constant and Einstein's constant, in the five-dimensional bulk.

With this choice, (8) acquire the following explicit form:

$$-e^{-2(f+h)}\left[2f_{,44} + \left(f_{,4}\right)^{2}\right] + 3\left[h_{,55} + 2\left(h_{,5}\right)^{2}\right] + \Lambda = \kappa P;$$

$$3e^{-2(f+h)}\left(f_{,4}\right)^{2} - 3\left[h_{,55} + 2\left(h_{,5}\right)^{2}\right] - \Lambda = \kappa\rho; \tag{9}$$

$$-3e^{-2(f+h)}\left[f_{,44} + \left(f_{,4}\right)^{2}\right] + 6\left(h_{,5}\right)^{2} + \Lambda = \kappa P;$$

where $f_{,4}$ is the derivative with respect to the conformal time, τ. From the first and the third equations, one may write down a geometric relation between the metric functions f and h:

$$e^{-2f}\left[f_{,44} + 2\left(f_{,4}\right)^{2}\right] + 3e^{2h}h_{,55} = 0. \tag{10}$$

The variables being separated, one can impose

$$e^{2h}h_{,55} = \omega^{2}, \tag{11}$$

and come to the following solutions [4]:

$$h(w) = \ln\left[\frac{\omega}{Q_0}\cosh(Q_0 w)\right], \tag{12a}$$

$$f(t) = \frac{1}{3}\ln\left[\frac{b}{2\omega}\sin(3\omega t)\right]. \tag{12b}$$

The constant Q_0 can be related to the cosmological constant of the AdS_5 bulk (which also fixes the AdS_5 scale) by

$$Q_0^2 = -\frac{\Lambda}{6} \sim \frac{1}{L^2},$$

and the parameter ω is proportional to the absolute value of the cosmological constant on the visible brane, as in [6],

$$\omega = \sqrt{\frac{|\Omega|}{3}}.$$

Finally, in order to understand the significance of the parameter b, we point out the fact that the scale function corresponding to (12b),

$$a(t) = e^{f(t)} = \left[\frac{b}{2\omega}\sin(3\omega t)\right]^{1/3}, \tag{13}$$

is actually describing a periodic Universe, with a finite-time cosmological singularity, for

$$3\omega t_n = n\pi.$$

When the Hubble's rate defined as $H = \dot{f}$ vanishes, for

$$3\omega t_k = (2k+1)\pi/2,$$

it occurs an instantaneous Minkowski-like phase, of maximum scale function a_0, to which the parameter b can be associated, by

$$b = 2\omega a_0^3. \tag{14}$$

The Quantum Analysis

As it is well-known, the quantum cosmology, in its traditional formulation, is based on the celebrated Wheeler–De Witt equation [4]

$$\mathbf{H}\psi = 0, \tag{15}$$

where the only dynamical degree of freedom is the radius of the Universe, a, and ψ is the wave function of the Universe.

The geometro-dynamical analysis on the class of solutions

$$ds_5^2 = e^{2h(w)}\left[e^{2f(t)}(d\vec{x})^2 - (dt)^2\right] + (dw)^2,$$

where $w \in \mathbb{R}$ stands for the local coordinate along the fifth dimension and the warped function $h\,(w)$ does specifically read (12a), is leading us to the following Hamiltonian-constraint-like equation

$$\mathbf{H} = \dot{a}^2 + V(a) = 0,$$

with the effective potential

$$\boldsymbol{V} = \omega^2 a^2 - \frac{b^2}{4a^4} = \omega^2 a^2\left[1 - \left(\frac{a_0}{a}\right)^6\right],$$

for $a(t)$ given by (13).

This suggests the definition of the "true" Lagrangian (not density or something else)

$$\mathbf{L}[a,\dot{a}] = \ell_0[\dot{a}^2 - \boldsymbol{V}(a)]\,,$$

where ℓ_0 is the inverse of the energy scale constant, fixing the characteristic length which comes from the fifth dimension, and the action functional with respect to the Universe scale-function reads, as usual,

$$\mathbf{S}[a] = \int \mathbf{L}[a,\dot{a}]dt.$$

The corresponding canonic conjugate momentum

$$\mathrm{p} = \frac{\partial \mathbf{L}}{\partial \dot{a}} = 2\ell_0\dot{a}\,,$$

together with the above Lagrangian, are leading, through the traditional canonic transformation

$$\mathbf{H} = \mathrm{p}\dot{a} - \mathbf{L}\,,$$

to the associated Hamiltonian

$$\mathbf{H} = \frac{1}{4\ell_0}\mathrm{p}^2 + \ell_0 \boldsymbol{V}(a). \tag{17}$$

Treating $\hat{a}$ and $\hat{\mathrm{p}}$ as operators with standard commutation relations, in the coordinate representation,

$$\mathrm{p} \to \hat{\mathrm{p}} = -i\frac{\partial}{\partial a},$$

the WDW equation (15), with the potential (16), explicitly reads

$$\frac{d^2\psi}{da^2} - W(a)\psi = 0, \tag{18}$$

with

$$W(a) \equiv W_0 a^2 \left[1 - \left(\frac{a_0}{a}\right)^6\right], \tag{19}$$

where $W_0 = 4\ell_0^2\omega^2$ is the characteristic amplitude of the reduced potential $W(a)$.

For small a values, (18) acquires the following form:

$$\frac{d^2\psi}{da^2} + \frac{d^2}{a^4}\psi = 0, \tag{20}$$

where $d^2 \equiv W_0 a_0^6$, its solutions being expressed in terms of the Bessel functions as

$$\psi(a) = \sqrt{a}\cdot \mathcal{J}_{\pm 1/2}\left(\frac{d}{a}\right) = a\sqrt{\frac{2}{\pi d}}\cdot\left\{\sin\left(\frac{d}{a}\right), \cos\left(\frac{d}{a}\right)\right\}. \tag{21}$$

Now, one may apply the Residue Theorem to compute the integral

$$\int_{\boldsymbol{R}} \psi^2 da = \frac{2}{\pi d}\int_{\boldsymbol{R}} a^2 \sin^2\left(\frac{d}{a}\right) da = \frac{8}{3}W_0 a_0^6.$$

In the opposite situation, when one is able to neglect the $\sim 1/a^4$ contribution in (19), the solution of the corresponding equation,

$$\frac{d^2\psi}{da^2} - W_0 a^2\psi = 0,$$

contains the Bessel function $Z_{\pm 1/4}$ of imaginary variable, so that

$$\psi = \sqrt{a}\cdot Z_{\pm 1/4}\left(\pm\frac{i}{2}\sqrt{W_0 a^2}\right).$$

Let us turn now to the time-evolving Schrödinger version of the mini-superspace WDW equation (15), namely

$$-\frac{1}{4\ell_0}\frac{\partial^2\psi}{\partial a^2} + \ell_0 V(a)\psi = i\frac{\partial\psi}{\partial t}. \tag{22}$$

For stationary states,

$$\psi_E(a,t) = \psi_E(a)e^{-iEt}, \tag{23}$$

the amplitude functions ψ_E must be solutions of the Schrödinger equation

$$\frac{d^2\psi_E}{da^2} + 4\ell_0(E - \ell_0 V)\psi_E = 0\,,$$

which can be written into the physically dimensionless form

$$\frac{d^2\psi_E}{da^2} + (\epsilon - W(a))\psi_E = 0, \tag{24}$$

where we have introduced the notations (19) and

$$\epsilon \equiv 4\ell_0 E. \tag{25}$$

Even though it is not possible to write down the analytical solution for the general Schrödinger-type equation (24), written as

$$\frac{d^2y}{da^2} + S(a)y = 0,$$

where, in our case,

$$S(a) = \epsilon - W_0 a^2 + \frac{d^2}{a^4},$$

one can study different properties, within theWKB approximation, valid for some ranges of the parameters. Thus, following the theory developed in [7], for $S(a) > 0$ and

$$-\frac{1}{4}\frac{S''}{S^2} + \frac{5}{16}\frac{(S')^2}{S^3} \ll 1$$

the solution acquires the periodic form

$$y(a) \sim \frac{C}{[S(a)]^{1/4}} \sin g(a)\,,$$

where the phase

$$g(a) = K + \int_{x_0}^{a} [S(x)]^{1/2}\, dx$$

gives the number of zeros, a_k, up to a, for $g(a_k) = k\pi$.

In the analyzis of (24), we firstly deal with small a values, for which it becomes

$$\frac{d^2\psi}{da^2} + \left(\epsilon + \frac{d^2}{a^4}\right)\psi = 0. \tag{26}$$

With the change of function

$$\psi = \sqrt{a}\varphi(a), \tag{27}$$

the above equation reads

$$\frac{d^2\varphi}{da^2} + \frac{1}{a}\frac{d\varphi}{da} + \left(\epsilon - \frac{1}{4a^2} + \frac{d^2}{a^4}\right)\varphi = 0, \tag{28}$$

or, in terms of the new variable $\xi = (1 + a^2)/(1 - a^2)$,

$$\frac{d^2\varphi}{d\xi^2} + \frac{2\xi}{\xi^2 - 1}\frac{d\varphi}{d\xi} + $$
$$+ \frac{1}{(\xi^2 - 1)^3}\left[\epsilon + d^2 + \frac{1}{4} - 2(\epsilon - d^2)\xi + (\epsilon + d^2 - \frac{1}{4})\xi^2\right]\varphi = 0. \tag{29}$$

By comparing the last expression with the Heun equation [6]

$$\frac{dy^2}{dz^2} - \frac{1}{(z^2 - 1)^2}[\alpha + 2z + \alpha z^2 - 2z^3]\frac{dy}{dz} + $$
$$+ \frac{1}{(z^2 - 1)^3}[\delta + (2\alpha + \gamma)z + \beta z^2]y = 0, \tag{30}$$

one may identify the function φ as being the Heun Double Confluent function, HeunD [8], of variable $\xi = (a^2 - 1)/(a^2 + 1)$ and parameters

$$\alpha = 0, \qquad \beta = \epsilon + d^2 - \frac{1}{4}, \qquad \gamma = -2(\epsilon - d^2), \qquad \delta = \epsilon + d^2 + \frac{1}{4}. \tag{31}$$

As for large a-values, once we neglect the $\sim 1/a^4$ contribution, the resulting equation

$$\frac{d^2\psi}{da^2} + (\epsilon - W_0 a^2)\psi = 0 \tag{32}$$

is satisfied by the Hermite associated functions [9]

$$\psi_n(a) = C_n \exp\left(-\frac{1}{2}\sqrt{W_0}a^2\right) H_n\left(W_0^{1/4}a\right), \tag{33}$$

with the energy spectrum

$$\epsilon_n = (2n+1)\sqrt{W_0} \Rightarrow E_n = \left(n + \frac{1}{2}\right)\omega = \left(n + \frac{1}{2}\right)\sqrt{\frac{|\Omega|}{3}}. \tag{34}$$

In the simplest non-trivial case, we can consider the mixture of the ground and first excited states,

$$\begin{aligned} \psi_0(a,t) &= \left(\frac{W_0}{\pi^2}\right)^{1/8} \exp\left[-\frac{1}{2}\sqrt{W_0}a^2\right] \exp\left[-\frac{i}{2}\omega t\right], \\ \psi_1(a,t) &= \sqrt{2}\left(\frac{W_0^3}{\pi^2}\right)^{1/8} a \exp\left[-\frac{1}{2}\sqrt{W_0}a^2\right] \exp\left[-\frac{i}{2}\omega t\right], \end{aligned} \tag{35}$$

with the respective initial probabilities 3/4 and 1/4, and it yields the periodic function

$$\begin{aligned} a(t) = \frac{1}{2}\left(\frac{W_0}{\pi^2}\right)^{1/4} \int_{-\infty}^{\infty} a e^{-\sqrt{W_0}a^2} \left[\sqrt{\frac{3}{2}} + W_0^{1/4} a e^{i\omega t}\right]\left[\sqrt{\frac{3}{2}} + W_0^{1/4} a e^{-i\omega t}\right] da = \\ = \frac{1}{4}\sqrt{\frac{3}{\ell_0 \omega}}\cos(\omega t). \end{aligned} \tag{36}$$

For large values of the quantum number n, so that $\epsilon \gg 4\sqrt{W_0}$, we write the solution of (32) as

$$\psi \sim \frac{1}{\sqrt{a}} M_{\lambda,\mu}\left(\sqrt{W_0}a^2\right), \tag{37}$$

where $M_{\lambda,\mu}$ is the Whittaker function with [7]

$$\lambda = \frac{1}{4}\frac{\epsilon}{\sqrt{W_0}}, \qquad \mu = \frac{1}{4},$$

and by employing the asymptotic representation (valid for large λ-values)

$$M_{\lambda,\mu}(\xi) \sim \frac{1}{\sqrt{\pi}}\Gamma(2\mu+1)\lambda^{-\mu-\frac{1}{4}}\xi^{\frac{1}{4}}\cos\left(2\sqrt{\lambda\xi} - \mu\pi - \frac{\pi}{4}\right), \tag{38}$$

we get the "free particle" amplitude function

$$\psi \sim \frac{W_0^{3/8}}{\sqrt{\epsilon}} \sin\sqrt{\epsilon}\, a. \tag{39}$$

In this section we discuss the time-evolving Schrödinger version of the WDW equation for the five dimensional k = 0—FRW Universe, whose warp factor and scale function are respectively given by (12a) and (13).

Even though there is an active debate on how one is able to include in the theory a time derivative term in the frozen WDW equation, strategies for connecting it to the time depending Schrödinger equation have been proposed [10].

Our geometro-dynamical analysis is based on the effective potential (16) for which the Schrödinger equation written for stationary states is (24). Such an equation does not have an analytical closed form solution, for W(a) given by (19). However, for specific ranges of the parameters, one may study different properties, as for example the density of nodes of the corresponding wave functions, using the WKB approach developed in [7]. For small a values, we get the wave function is expressed in terms of the Heun Double Confluent function, of parameters (31).

Even though these functions have been intensively worked out in the last years, in situations relevant to physics, chemistry and engineering [11], there are unsolved problems when dealing with the general expressions, especially related to their normalization or series expansions. However, physically reasonable solutions describing the stationary spectra of comptonization in a photon flux along the frequency axis, written in terms of the HeunD function and its derivative, have been discussed in [12].

In the opposite situation (a large), the results can be put in a more transparent form, the corresponding Hermite associated functions leading, for the mixture of the ground and first excited states, to the periodic function (36). Finally, for from the asymptotic representation of the Whittaker function, (38), valid for large values of the quantum number n, it pops up to the "free particle" amplitude (39).

IV.4. On a Schrödinger-like equation with some special potential

Conclusions Chapter IV

Section IV.3. is devoted to the time-evolving Schrödinger version of the Wheeler-DeWitt equation, written for the five dimensional warped k = 0—FRW Universe. For small values of the cosmological scale factor, a, the wave function of the Universe is expressed in terms of the Heun Double Confluent functions, which have been intensively worked out in the last years. As expected, for large a's, one gets the well-known Hermite associated functions.Within the semiclassical approximation, valid for large n, the asymptotic representation of the Whittaker functions leads to the "free particle" behavior.

These results are the very subject of the paper [13].

Secțion IV.4. is devoted to the time-evolving one-dimensional Schrödinger equation for stationary state with some special potential, version of the Wheeler-De Witt equation. For small values of the cosmological scale factor the wave function of the Universe is expressed in terms of the Heun Double Confluent functions. For large values one gets the well-known Hermite associated functions.

These results are the very subject of the paper [14].

Selective bibliography Chapter IV

[1] L. Randall, R. Sundrum, *Large Mass Hierarchy from a Small Extra Dimension*, Phys. Rev. Lett., Vol. 83, Issue 17, p. 3370, (1999).

[2] D. Bazeia, F.A. Brito, L. Losano, *Scalar fields, bent Branes, and RG flow*, J. High Energy Phys., Vol. 2006, Issue 11, p. 64, (2006).

[3] M. Gremm, *Four-dimensional gravity on a thick domain wall*, Phys. Lett. B, Vol. 478, p. 434, (2000).

[4] J.A. Wheeler, *Superspace and the nature of quantum geometrodynamics*, Batelle Rencontres. Benjamin, New York, (1968).

[5] M.A. Dariescu, C. Dariescu, *Robertson-Walker Branes with massless scalars and cosmological term*, Astropart. Phys.,Vol. 34, Issue 2, p. 116, (2010).

[6] R. Koley, J. Mitra, S. SenGupta, *Fermion localization in a generalized Randall-Sundrum model*, Phys. Rev. D, Vol. 79, Issue 4, p. 1902, (2009).

[7] E.R. Arriola, A. Zarzo, J.S. Dehesa, *Spectral properties of the biconfluent Heun equation*, J. Comput. Appl. Math., Vol. 37, p. 161, (1991).

[8] F.M. Arscott, *Heun's equation*, in: Ronveaux, A. (ed.) Heun's Differential Equations, Oxford University Press, Oxford (1995).

[9] I.S. Gradshteyn, I.M. Ryzhik, *Table of Integrals, Series and Products*, 4th edn Academic, New York, (1965).

[10] M. Pavsic, *Condensed Wheeler-DeWitt Equation in Five Dimensions and Modified QEDmatter physics*, Preprint arXiv: 1207.4594 [gr-qc], (2012).

[11] P.P. Fiziev, D.R. Staicova, *Solving systems of transcendental equations involving the Heun functions*, Preprint arXiv: 1201.0017v1 [hep-ph], (2011).

[12] A.E. Dubinov, *Exact stationary solution of the Kompaneets kinetic equation*, Tech. Phys. Lett., Vol. 35, No. 3, p. 260, (2009).

[13] M.A. Dariescu, C.Dariescu, C.Cretu, *The Quantum Treatment of the 5D-Warped Friedmann–Robertson–Walker Universe in Schrödinger Picture*, International Journal of

[14] C.Crețu, *On a Schrödinger-like equation with some special potential*, Buletinul Institutului Politehnic din Iaşi, Tomul LIX(LXIII), Fasc. 3, p. 67, (2013).

General conclusions

Since every chapter contains a section of conclusions in which the main results are highlighted by making comparisons with the works of other authors too, in this section we are going to synthetize a few general conclusions that we consider outstanding and which represent the subject of the works published in magazines ISI rated and in magazines acknowledged by CNCSIS.

The results can be synthetized in the following way:

1. Starting from a class of planary symmetric metrics in null-coordinate formulation I obtained the Killing vectors. Using a proper change of coordinates, I calculated the Lie derivative of metric

$$ds^2 = e^{2f(z,t)}(dx^2 + dy^2) + dz^2 - dt^2 \ .$$

By respecting the conditions of integrability I solved the corresponding Killing equations.

[C. Crețu, C. Dariescu, *On the Isometry Group for a Class of Planary Symmetric Metrics in Null-coordinate Formulation* , presentation at TIM14 Physics Conference, Univ. de Vest, Timișoara, (2014).]

2. Using an idea of R.G. Beil, we modify the Minkowski metric of the space in which a charged particle moves according to Lorentz equation. The modification is such that, with the new metric and in the new space, the particle moves on a geodesic. The process of obtaining the metric appears like a gauge transformation. The dependence on the point (x) and speed (v) of the scale factor b is transmitted to vector B_μ and even further to the metric $\bar{g}_{\mu\nu}$. Being dependent on point and speed, $\bar{g}_{\mu\nu}(x,v)$ is a generalized Lagrange metric.

[C.Crețu, *Electrodynamics from modified Schwarzschild metric* , presentation at TIM13 Physics Conference, Univ. de Vest, Timișoara, (2013).]

3. For the relativist bosons evolving in static orthogonal electric and magnetic fields, I obtained the wave functions and the energetic spectra, in a good concordance with the results of other authors. Within the thermodynamic study, the partition function expressed by the Euler and Riemann generalized functions lead to the main thermodynamic sizes of the magnetization and susceptibility. The equation of state contains the term of Hall type and additional contributions which characterize the ultra-relativist particles. In particular cases, for a certain domain of parameters, the equation of state leads to classical thermodynamic results (high temperatures) and to quantic effects (low temperatures) in accordance with Nernst theorem.

[M.A. Dariescu, O. Buhucianu , C. Crețu, *The Lorentz-invariant U(1)-gauge theory of scalars in static external fields and thermal properties*, Buletinul Institutului Politehnic din Iaşi, Tomul LVIII(LIX), Fasc. 2, p. 43, (2012).]

4. Starting with the wave function characterizing massless fermions evolving in orthogonal electric and magnetic fields, written in terms of Heun Biconfluent functions, I analyzed some physically interesting cases. When the HeunB function truncates toa polynomial form, one may easily compute the essential components of the conserved current density. For a vanishing electric field, I obtained the familiar Hermite associated functions and I discussed

the current dependence on the sample width. In the opposite case, corresponding to an electric static field alone, one has to deal with HeunB functions of complex variable and parameters.
[Dariescu, C. Dariescu, C. Creţu, O. Buhucianu, *Analytic Study of Fermions in Graphene; Heun Functions and Beyond*, Rom.Journ.Phys.,Vol. 58, Nos.7-8, p. 706, Bucharest, (2013).]

5. I analyzed the time-evolving Schrödinger version of the Wheeler–De Witt equation, written for the five dimensional warped k = 0-FRW Universe. For small values of the cosmological scale factor, a, the wave function of the Universe is expressed in terms of the Heun Double Confluent functions, which have been intensively worked out in the last years. As expected, for large a's, one gets the well-known Hermite associated functions.Within the semiclassical approximation, valid for large *n*, the asymptotic representation of the Whittaker functions leads to the "free particle" behavior.
[M.A.Dariescu, C.Dariescu, C.Creţu, *The Quantum Treatment of the 5D-Warped Friedmann–Robertson–Walker Universe in Schrödinger Picture*, International Journal of Theoretical Physics, Volume 52, Issue 4, p. 1345, (2013).]

6. I detailed the time-evolving one-dimensional Schrödinger equation for stationary state with some special potential, version of the Wheeler-De Witt equation. For small values of the cosmological scale factor the wave function of the Universe is expressed in terms of the Heun Double Confluent functions. For large values one gets the well-known Hermite associated functions.
[C.Creţu, *On a Schrödinger-like equation with some special potential*, Buletinul Institutului Politehnic din Iaşi, Tomul LIX(LXIII), Fasc. 3, p. 67, (2013).]

List of personal publications

A. 1. Publications in magazines ISI rated

1. M.A. Dariescu, C. Dariescu, C.Creţu, *The Quantum Treatment of the 5D-Warped Friedmann–Robertson–Walker Universe in Schrödinger Picture,* International Journal of Theoretical Physics, Volume 52, Issue 4, p. 1345, (2013).
(Impact factor upon author = 1,188 / 3 = 0,4 , Influence score = 0,182)

2. M.A. Dariescu, C. Dariescu, C.Creţu, O. Buhucianu, *Analytic Study of Fermions in Graphene; Heun Functions and Beyond,* Rom.Journ.Phys.,Vol. 58, Nos.7-8, p. 706, Bucharest, (2013).
(Impact factor upon author = 0,745 / 4 = 0,18 , Influence score = 0.124)

Total impact factor / author = 0,58
Total influence score = 0,306

A. 2. Publications in magazines of category B+

(Acknowledged CNCSIS)

1. M. A. Dariescu, O. Buhucianu, C. Creţu, *The Lorentz-invariant U(1)-gauge theory of scalars in static external fields and thermal properties,* Buletinul Institutului Politehnic din Iaşi, Tomul LVIII(LIX), Fasc. 2, p. 43, (2012).

2. C.Creţu, *On a Schrödinger-like equation with some special potential*, Buletinul Institutului Politehnic din Iaşi, Tomul LIX(LXIII), Fasc. 3, p. 67, (2013).

B. Works presented at international conferences

1. M.A. Dariescu, C.Dariescu, C.Creţu, *The Schrödinger-Poisson heuristic limit of the Klein-Gordon-Maxwell-system and its S-Eigenmode solution for a spherically trapped particle,* ICPAM 2012, International Conference on Physics of Advanced Materials, Universitatea "Al.I.Cuza" din Iași, 20-23 sep. (2012).

2. C.Creţu, *Electrodinamică dintr-o metrică Schwarzshild modificată,* TIM 2013, Physics Conference – Physics without frontiers, Universitatea de Vest, Timișoara, 21-23 nov. (2013).

3. C. Creţu, C. Dariescu, *On the Isometry Group for a Class of Planary Symmetric Metrics in Null-coordinate Formulation*, TIM 2014, Physics Conference – Physics without frontiers, Universitatea de Vest, Timişoara, 20-22 nov. (2014).

C. Works presented at national conferences

1. C.Crețu, *Considerații asupra unei ecuații de tip Schrödinger cu un potențial special*, Conferința Națională de Fizică Aplicată, Universitatea Tehnică „Gh.Asachi" din Iași, 23-24 mai, (2013).