„Alexandru Ioan Cuza" University of Iași
Faculty of Mathematics

Oblique reflection in stochastic models
(Reflexia oblică în modele stochastice)

Ph.D. thesis

by

Anouar M. GASSOUS

Scientific coordinator
Prof. dr. Aurel Rășcanu

Iași, 2013

UNIVERSITATEA „AL. I. CUZA”, IAŞI
RECTORATUL

D-lui/D-nei ....

Vă facem cunoscut faptul că, la data de 06.11.2013, ora 12:00, în Sala de Conferințe a Facultății de Matematică, domnul Anouar M. GASSOUS va susține în şedință publică teza de doctorat intitulată „OBLIQUE REFLECTION IN STOCHASTIC MODELS (REFLEXIA OBLICĂ ÎN MODELE STOCHASTICE)”, în vederea obținerii titlului ştiințific de Doctor în domeniul MATEMATICĂ.

Comisia de doctorat are următoarea componență:

Preşedinte:

Prof. univ. dr. CATĂLIN GEORGE LEFTER, Decan al Facultății de Matematică, Universitatea „Al. I. Cuza”, Iaşi

Conducător ştiințific:

Prof. univ. dr. AUREL RĂŞCANU, Universitatea „Al. I. Cuza”, Iaşi

Referenți:

Acad. VIOREL BARBU, Academia Română, Filiala Iași
Prof. univ.dr. OVIDIU CÂRJĂ, Universitatea „Al. I. Cuza”, Iaşi
Prof. univ. dr. NARCISA DUMITRIU, Universitatea Tehnică „Gh. Asachi”, Iaşi

Vă transmitem rezumatul tezei de doctorat şi vă invităm să participați la şedința de susținere publică a tezei.

Contents

Introduction

In this thesis we generalize known results from the theory of stochastic variational inequalities, forward and backward as well. As a leading direction for the entire study, our problems aim to the case when the reflection from the system is no more upon the normal direction, but it becomes a generalized one, which will be called *oblique subgradient*. Even a consistent part of the thesis is considered on the convex setup, we are also interested to analyze the problem of oblique reflection for deterministic systems considered in a non-convex framework. These kind of problems lead to generalized Skorohod problems and stochastic variational inequalities with oblique subgradients in non-convex domains.

The thesis is based on the papers [22], [23], [24], [25] and it is structured into five chapters and one Annex.

Since early sixties, research has paid increasing attention to the study of reflected stochastic differential equations (for short, SDE), the reflection process being approached in different ways. Skorohod, for instance, considers the problem of reflection for diffusion processes into a bounded domain (see, e.g., [52]). Tanaka focuses on the problem of reflecting boundary conditions into convex sets for SDEs (see [53]). In the early stages of research on the topic the trajectories of the system were reflected upon the normal direction but, in 1984 Lions & Sznitman, in the paper [30], study for the first time the following problem of oblique reflection in a domain:

$$(1)\qquad \begin{cases} dX_t + dK_t = f(t, X_t)\,dt + g(t, X_t)\,dB_t, \quad t > 0, \\ X_0 = x, \quad K_t = \displaystyle\int_0^t 1_{\{X_s \in Bd(E)\}} \gamma(X_s)\, d\updownarrow K \updownarrow_s, \end{cases}$$

where, for the bounded oblique reflection $\gamma \in \mathcal{C}^2\left(\mathbb{R}^d\right)$, there exists a positive constant ν such that $\langle \gamma(x), n(x) \rangle \geq \nu$, for every $x \in Bd(E)$, $n(x)$ being the unit outward normal vector. Many generalizations of the problem were considered, starting with the papers of Depuis & Ishi [19] and [20] or Słomiński [51]

A more general form of the Skorohod problem was introduced in 1981 by Rǎşcanu; he studied the multivalued SDEs with subdifferential operator, also called stochastic variational inequalities, consistent results being provided in 1997 by Asiminoaei & Rǎşcanu in [1], Barbu & Rǎşcanu in [4] and Bensoussan & Rǎşcanu in [5]. They prove the existence and uniqueness result for the case of stochastic variational differential systems involving subdifferential operators and, even more, they provide approximation and splitting-up schemes for this type of equations.

As the main objective of Chapter 2, we prove the existence and uniqueness of the solution for the following stochastic variational inequality

$$(2)\qquad \begin{cases} dX_t + H(X_t)\,\partial\varphi(X_t)\,(dt) \ni f(t, X_t)\,dt + g(t, X_t)\,dB_t, \qquad t > 0, \\ X_0 = x_0, \end{cases}$$

where B is a standard Brownian motion defined on a complete probability space and the new quantity $H(X)$ that appears acts on the set of subgradients; the product $H(X)\,\partial\varphi(X)$ will be called "*oblique subgradients*". For proving the existence, we will first solve the deterministic case, considering a generalized Skorohod problem with oblique reflection of the form

$$(3)\qquad \begin{cases} x\left(t\right)+\int_{0}^{t}H\left(x\left(s\right)\right)dk\left(s\right)=x_{0}+\int_{0}^{t}f\left(s,x\left(s\right)\right)ds+m\left(t\right), & t\geq 0,\\ dk\left(s\right)\in\partial\varphi\left(x\left(s\right)\right)\left(ds\right), \end{cases}$$

where the singular input $m:\mathbb{R}_{+}\rightarrow\mathbb{R}^{d}$ is a continuous function.

It worth mentioning that, until now, when dealing with BSVIs, the reflection was made upon the normal direction at the frontier of the domain and it was caused by the presence of the subdifferential operator of a convex lower semicontinuous function. In the main study of Chapter 3, we prove the existence and uniqueness of the solution for the more generalized BSVI with oblique subgradients

$$\begin{cases} -dY_{t}+H\left(t,Y_{t}\right)\partial\varphi\left(Y_{t}\right)\left(dt\right)\ni F\left(t,Y_{t},Z_{t}\right)dt-Z_{t}dB_{t}, & t\in\left[0,T\right],\\ Y_{T}=\eta, \end{cases}$$

where B is a standard Brownian motion defined on a complete probability space, F is the generator function and the random variable η is the terminal data. We will split our problem into two new ones. For the situation when we have only a time dependence for the matrix H we obtain the existence of a strong solution, together with the existence of an absolutely continuous feedback-subgradient process. However, for the general case of a state dependence for H we will use tightness criteria in order to get a solution for the equation. We will put our problem into a Markovian framework, which permits also, via the non-linear Feynman-Kaç representation formula, to approach the viscosity solutions for semilinear parabolic PDEs. The problem consists in answering in which sense can we take the limit in the sequence $\{(Y^{n},Z^{n},U^{n})\}_{n}$, given by the solutions of the approximating equations. We have to prove that it is tight in a certain topology. Even the $S-$topology introduced by Jakubowski in [27] (and used for similar setups by Boufoussi and Casteren [9] or LeJai [29]) seems suitable for our context, the regularity of the subgradient process given by the approximating equation as part of its solution permits us to show a convergence in the sense of the Meyer-Zheng topology.

In Chapter 4 the research focuses on the multivalued differential equations (also called variational inequalities), driven by the Fréchet subdifferential operator $\partial^{-}\varphi$ and perturbed by the matrix $H(X)$, where solution borders a domain not necessarily convex.. The non-convex problem, without the term $H(X)$, has been solved by Marino & Tosques [32] or Rossi & Savaré [46] and [47]. The reflected problem ($\varphi=I_{E}$) with singular input dm/dt have been treated by Lions and Sznitman in [30] or Saisho in [49].

In Chapter 5, we consider another type of non-convex Skorohod problem. More precisely, the equation, this time driven by the Clarke subdifferential operator, is given by

$$\begin{cases} dx\left(t\right)+\partial\varphi\left(x\left(t\right)\right)dt+\partial_{C}\phi\left(x\left(t\right)\right)dt+G(t,x\left(t\right))dt\ni dm\left(t\right), & t\in\left[0,T\right],\\ x\left(0\right)=x_{0}. \end{cases}$$

We provide existence and uniqueness results for the solution of the above equation, then we extend it to the stochastic case.

The last chapter, groups together, under the form of an *Annex,* some important results and instruments which allow us to obtain the tasks established along the thesis.

1 Preliminaries

1.1 Elements of convex and non-convex analysis

Convex functions

If K is a convex subset of $\mathbb{R}^d$, then $I_K : \mathbb{R}^d \to (-\infty, +\infty]$, given by

$$I_K(x) = \begin{cases} 0, & \text{for } x \in K, \\ +\infty, & \text{for } x \in \mathbb{R}^d \setminus K \end{cases}$$

is a convex function called the convex indicator of K.

Definition 1 *A function $\varphi : \mathbb{R}^d \to (-\infty, +\infty]$ is lower semicontinuous (l.s.c., for short) in $x \in \mathbb{R}^d$ if*

$$\varphi(x) \leq \liminf_{y \to x} \varphi(y).$$

We say that φ is lower semicontinuous if it is lower semicontinuous in every $x \in \mathbb{R}^d$.

Denote by $\partial\varphi$ the subdifferential operator of φ:

$$\partial\varphi(x) \overset{def}{=} \left\{ \hat{x} \in \mathbb{R}^d : \langle \hat{x}, y - x \rangle + \varphi(x) \leq \varphi(y), \text{ for all } y \in \mathbb{R}^d \right\},$$

and by $\nabla\varphi_\varepsilon$ we denote the gradient of the Yosida's regularization φ_ε of the convex lower semicontinuous function φ, that is

$$\varphi_\varepsilon(x) = \inf\{\frac{1}{2\varepsilon}|z - x|^2 + \varphi(z) : z \in \mathbb{R}^d\} = \frac{1}{2\varepsilon}|x - J_\varepsilon x|^2 + \varphi(J_\varepsilon x),$$

where $J_\varepsilon x = x - \varepsilon \nabla\varphi_\varepsilon(x)$.

(ρ, γ) **−semiconvex functions**

Let consider the Euclidian space $\mathbb{R}^d$ with $\langle \cdot, \cdot \rangle$ the inner product and $|\cdot|$ the induced norm.

Definition 2 *Let $\gamma \geq 0$. A set E is γ−semiconvex if for all $x \in Bd(E)$ there exists $\hat{x} \in \mathbb{R}^d \setminus \{0\}$ such that*

$$\langle \hat{x}, y - x \rangle \leq \gamma |\hat{x}| |y - x|^2, \quad \forall y \in E.$$

Let $\varphi : \mathbb{R}^d \to (-\infty, +\infty]$, with $Dom(\varphi) = \left\{ v \in \mathbb{R}^d : \varphi(v) < +\infty \right\}$.

Definition 3 *Define the Fréchet subdifferential of φ at $u \in \mathbb{R}^d$ by*

$$\partial^-\varphi(u) = \{u^* \in \mathbb{R}^d : \quad \liminf_{v \to u; v \neq u} \frac{\varphi(v) - \varphi(u) - \langle u^*, v - u \rangle}{|v - u|} \geq 0\},$$

if $u \in Dom(\varphi)$, and $\partial^-\varphi(u) = \emptyset$ if $u \notin Dom(\varphi)$.

Proposition 4 *Consider $u \in \mathbb{R}^d$ and let ($J_\varepsilon u \overset{def}{=}$) u_ε be a locally minimizing point (if there exists) of the function*

$$v \longrightarrow \frac{1}{2\varepsilon}|u-v|^2 + \varphi(v) : \mathbb{R}^d \to (-\infty, +\infty].$$

Then

$$u_\varepsilon \in Dom\left(\partial^- \varphi\right) \quad \text{and} \quad \left(A_\varepsilon(u) \overset{def}{=}\right) \frac{1}{\varepsilon}(u - u_\varepsilon) \in \partial^- \varphi(u_\varepsilon).$$

Definition 5 *Let $\rho, \gamma > 0$. The function $\varphi : \mathbb{R}^d \to (-\infty, +\infty]$ is a $(\rho, \gamma)-$semiconvex function if $\overline{Dom(\varphi)}$ is $\gamma-$semiconvex, $Dom(\partial^- \varphi) \neq \emptyset$ and, for every $(u, u^*) \in \partial^- \varphi$ and $v \in \mathbb{R}^d$:*

$$\langle u^*, v-u\rangle + \varphi(u) \leq \varphi(v) + (\rho + \gamma|u^*|)|v-u|^2.$$

Clarke subdifferential operator

Definition 6 *Let $\mathbb{V}$ be a Banach space, $f : \mathbb{V} \to \mathbb{R}$ be Lipschitz near a given point $x_0 \in \mathbb{V}$, and v a vector in $\mathbb{V}$. We denote by $f^o(x_0, v)$ the generalized directional derivative of f at x_0 in the direction v, defined:*

$$f^o(x_0; v) \overset{def}{=} \limsup_{y \to x_0, \delta \downarrow 0} \frac{f(y + \delta v) - f(y)}{\delta},$$

then the Clarke subdifferential of f at x_0 is defined as follows:

$$\partial_C f(x) \overset{def}{=} \{\xi \in V^* : \langle \xi, v\rangle \leq f^o(x_0; v) \text{ for all } v \in \mathbb{V}\}.$$

1.2 Elements of stochastic analysis

This section recalls some basic elements of stochastic analysis: stochastic bases, filtrations, conditional expectation of a random variable, stochastic processes, stochastic integral, Itô's formula. Important and useful results concerning stochastic differential equations and backward stochastic differential equations are also provided.

2 Stochastic variational inequalities with oblique subgradients

In this chapter we study the existence and uniqueness of the solution for the stochastic variational inequality with oblique subgradients of the following form

$$\begin{cases} dX_t + H(X_t)\partial\varphi(X_t)(dt) \ni f(t, X_t)dt + g(t, X_t)dB_t, \quad t \geq 0, \\ X_0 = x \in \overline{Dom(\varphi)}. \end{cases}$$

The existence result is based on a deterministic approach: a differential system with singular input is first analyzed. After this we continue the study in the stochastic setup.

2.1 A generalized Skorohod problem with oblique reflection

2.1.1 Formulation of the problem and main assumptions

We consider the deterministic generalized convex Skorohod problem with oblique subgradients:

$$\text{(4)} \qquad \begin{cases} dx(t) + H(x(t))\,\partial\varphi(x(t))(dt) \ni dm(t), & t > 0, \\ x(0) = x_0, \end{cases}$$

where

$$\text{(5)} \qquad \varphi : \mathbb{R}^d \to \left]-\infty, +\infty\right] \text{ is a proper convex l.s.c. function}$$

and

$$\text{(6)} \qquad \begin{cases} (i) & x_0 \in Dom(\varphi) \overset{def}{=} \{x \in \mathbb{R}^d : \varphi(x) < \infty\}, \\ (ii) & m \in C\left(\mathbb{R}_+; \mathbb{R}^d\right), \quad m(0) = 0. \end{cases}$$

$H = (h_{i,j})_{d\times d} \in C_b^2\left(\mathbb{R}^d; \mathbb{R}^{d\times d}\right)$ is a matrix, such that for all $x \in \mathbb{R}^d$,

$$\text{(7)} \qquad \begin{cases} (i) & h_{i,j}(x) = h_{j,i}(x), \quad \text{for every } i, j \in \overline{1, d}, \\ (ii) & \dfrac{1}{c}|u|^2 \le \langle H(x)u, u\rangle \le c|u|^2, \quad \forall\, u \in \mathbb{R}^d \text{ (for some } c \ge 1). \end{cases}$$

Let $[H(x)]^{-1}$ be the inverse matrix of $H(x)$. Then $[H(x)]^{-1}$ has the same properties (7) as $H(x)$. Denote

$$b = \sup_{x,y\in\mathbb{R}^d,\ x\neq y} \frac{|H(x) - H(y)|}{|x-y|} + \sup_{x,y\in\mathbb{R}^d,\ x\neq y} \frac{|\,[H(x)]^{-1} - [H(y)]^{-1}\,|}{|x-y|},$$

where $|H(x)| \overset{def}{=} \left(\sum_{i,j=1}^d |h_{i,j}(x)|^2\right)^{1/2}$.

Remark 7 *A vector ν_x associated to $x \in Bd(E)$ (we denote by $Bd(E)$ the boundary of the set E) is called external direction if there exists $\rho_0 > 0$ such that $x + \rho\nu_x \notin E$ for all $0 < \rho \le \rho_0$. In this case, if there exist $c' > 0$ and $n_x \in N_E(x)$, $|n_x| = 1$, such that $\langle n_x, \nu_x\rangle \ge c'$, then we have the representation*

$$\nu_x = M(x)\, n_x\ , \textit{ for all } x \in Bd(E),$$

where the symmetric matrix

$$\text{(8)} \qquad M(x) = \langle \nu_x, n_x\rangle I_{d\times d} - \nu_x \otimes n_x - n_x \otimes \nu_x + \frac{2}{\langle \nu_x, n_x\rangle}\nu_x \otimes \nu_x\ ,$$

We shall call *oblique reflection* directions of the form $\nu_x = H(x)\, n_x$, with $x \in Bd(E)$, where $n_x \in N_E(x)$.

If $E = \overline{E} \subset \mathbb{R}^d$ and $E^c = \mathbb{R}^d \setminus E$, then we denote, for $\varepsilon > 0$,

$$E_\varepsilon = \{x \in E : dist(x, E^c) \geq \varepsilon\} = \overline{\{x \in E : B(x, \varepsilon) \subset E\}}$$

the $\varepsilon-$interior of E.

We impose the following supplementary assumptions:

$$\text{(9)} \qquad \begin{cases} (i) & D = Dom(\varphi) \text{ is a closed subset of } \mathbb{R}^d, \\ (ii) & \exists\, r_0 > 0,\ D_{r_0} \neq \emptyset \quad \text{and} \quad h_0 = \sup_{z \in D} dist(z, D_{r_0}) < \infty, \\ (iii) & \exists\, L \geq 0, \text{ such that } |\varphi(x) - \varphi(y)| \leq L + L\,|x - y|\,, \\ & \hfill \text{for all } x, y \in D. \end{cases}$$

Definition 8 *Given two functions $x, k : \mathbb{R}_+ \to \mathbb{R}^d$, we say that $dk(t) \in \partial\varphi(x(t))(dt)$ if*

$$\begin{aligned} &(a) \quad x, k : \mathbb{R}_+ \to \mathbb{R}^d \text{ are continuous,} \\ &(b) \quad x(t) \in \overline{Dom(\varphi)}, \\ &(c) \quad k \in BV_{loc}\left([0, +\infty[; \mathbb{R}^d\right), k(0) = 0, \\ &(d) \quad \int_s^t \langle y(r) - x(r), dk(r)\rangle + \int_s^t \varphi(x(r))\, dr \leq \int_s^t \varphi(y(r))\, dr, \\ &\qquad\qquad \text{for all } 0 \leq s \leq t \leq T \text{ and } y \in C\left([0, T]; \mathbb{R}^d\right). \end{aligned}$$

We state that

Definition 9 *A pair of functions (x, k) is a solution of the Skorohod problem with $H-$oblique subgradients (4) (and we write $(x, k) \in \mathcal{SP}(H\partial\varphi; x_0, m)$) if $x, k : \mathbb{R}_+ \to \mathbb{R}^d$ are continuous functions and*

$$\text{(10)} \qquad \begin{cases} (i) & x(t) + \displaystyle\int_0^t H(x(r))\, dk(r) = x_0 + m(t)\,, \quad \forall\, t \geq 0, \\ (ii) & dk(r) \in \partial\varphi(x(r))(dr)\,. \end{cases}$$

By a direct calculus, from the above Definitions, we obtain the following inequality, very important for the uniqueness of the solution.

If $(x, k) \in \mathcal{SP}(H\partial\varphi; x_0, m)$ and $(\hat{x}, \hat{k}) \in \mathcal{SP}(H\partial\varphi; \hat{x}_0, \hat{m})$, then for all $0 \leq s \leq t$:

$$\text{(11)} \qquad \int_s^t \left\langle x(r) - \hat{x}(r)\,, dk(r) - d\hat{k}(r)\right\rangle \geq 0.$$

Proposition 10 *If $(x, k) \in \mathcal{SP}(H\partial\varphi; x_0, m)$ then, under assumptions (6), (7), (5) and (9) there exists a constant $C_T(\|m\|_T) = C(T, \|m\|_T, b, c, r_0, h_0)$, increasing function with respect to $\|m\|_T$, such that, for all $0 \leq s \leq t \leq T$,*

$$\text{(12)} \qquad \begin{aligned} &(a) \quad ||x||_T + \updownarrow k \updownarrow_T \leq C_T(||m||_T), \\ &(b) \quad |x(t) - x(s)| + \updownarrow k \updownarrow_t - \updownarrow k \updownarrow_s \leq C_T(||m||_T) \times \sqrt{t - s + \mathbf{m}_m(t - s)}, \end{aligned}$$

where $\mathbf{m}_m$ represents the modulus of continuity of the continuous function m.

We renounce now at the restriction that the function f is identically 0 and we consider the equation written under differential form

$$\text{(13)} \qquad \begin{cases} dx(t) + H(x(t))\,\partial\varphi(x(t))(dt) \ni f(t, x(t))\,dt + dm(t), & t > 0, \\ x(0) = x_0, \end{cases}$$

where

$$\text{(14)} \qquad \begin{array}{ll} (i) & (t,x) \longmapsto f(t,x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \text{ is a Carathéodory function} \\ & \qquad \text{(i.e. measurable w.r. to } t \text{ and continuous w.r. to } x\text{)}, \\ (ii) & \displaystyle\int_0^T \left(f^{\#}(t)\right)^2 dt < \infty, \quad \text{where} \quad f^{\#}(t) = \sup_{x \in Dom(\varphi)} |f(t,x)|. \end{array}$$

2.1.2 The existence result

We are now able to formulate the main results of this chapter.

Theorem 11 *Let the assumptions (6), (7), (5), (9) and (14) be satisfied. Then the differential equation (13) has at least one solution in the sense of Definition 9, i.e.* $x, k : \mathbb{R}_+ \to \mathbb{R}^d$ *are continuous functions and*

$$\text{(15)} \qquad \begin{cases} (j) & x(t) + \displaystyle\int_0^t H(x(r))\,dk(r) = x_0 + \int_0^t f(r, x(r))\,dr + m(t), \quad \forall\, t \geq 0, \\ (jj) & dk(r) \in \partial\varphi(x(r))(dr). \end{cases}$$

2.1.3 The uniqueness result

In the next step we will impose additional assumptions in order to have the uniqueness of the solution for Eq.(13).

Proposition 12 *Let the assumptions (7), (6), (5), (9) and (14) be satisfied. Assume also that there exists* $\mu \in L^1_{loc}(\mathbb{R}_+; \mathbb{R}_+)$ *such that, for all* $x, y \in \mathbb{R}^d$,

$$\text{(16)} \qquad |f(t,x) - f(t,y)| \leq \mu(t)\,|x - y|, \quad a.e.\ t \geq 0.$$

If $m \in BV_{loc}\left(\mathbb{R}_+; \mathbb{R}^d\right)$, *then the generalized convex Skorohod problem with oblique subgradients (4) admits a unique solution* (x, k) *in the space* $C(\mathbb{R}_+; \mathbb{R}^d) \times [C(\mathbb{R}_+; \mathbb{R}^d) \cap BV_{loc}(\mathbb{R}_+; \mathbb{R}^d)]$. *Moreover, if* (x, k) *and* $(\hat{x}, \hat{k})$ *are two solutions, corresponding to* m, *respectively* $\hat{m}$, *then*

$$\text{(17)} \qquad |x(t) - \hat{x}(t)| \leq C e^{CV(t)} \left[|x_0 - \hat{x}_0| + \updownarrow m - \hat{m} \updownarrow_t\right],$$

where $V(t) = \updownarrow x \updownarrow_t + \updownarrow \hat{x} \updownarrow_t + \updownarrow k \updownarrow_t + \updownarrow \hat{k} \updownarrow_t + \displaystyle\int_0^t \mu(r)\,dr$ *and* C *is a constant depending only on* b *and* c.

Proposition 13 *Under the assumptions of Proposition 12 and, for $m \in C^1(\mathbb{R}_+;\mathbb{R}^d)$, the solution $(x_\varepsilon)_{0<\varepsilon\leq 1}$ of the approximating equation*

$$(18)\qquad \begin{aligned} &x_\varepsilon(t) + \int_0^t H(x_\varepsilon(s))\,dk_\varepsilon(s) = x_0 + \int_0^t f(s,\pi_D(x_\varepsilon(s)))\,ds + m(t), \quad t\geq 0,\\ &dk_\varepsilon(s) = \nabla\varphi_\varepsilon(x_\varepsilon(s))\,ds, \end{aligned}$$

has the following properties:
• *for all $T>0$ there exists a constant C_T, independent of $\varepsilon,\delta \in]0,1]$, such that*

$$\begin{aligned} &(j) && \sup_{t\in[0,T]} |x_\varepsilon(t)|^2 + \sup\nolimits_{t\in[0,T]} |\varphi_\varepsilon(x_\varepsilon(t))| + \int_0^T |\nabla\varphi_\varepsilon(x_\varepsilon(s))|^2\,ds \leq C_T,\\ &(jj) && \updownarrow x_\varepsilon \updownarrow_{[s,t]} \leq C_T\sqrt{t-s}, \quad \textit{for all } 0\leq s\leq t\leq T,\\ &(jjj) && \|x_\varepsilon - x_\delta\|_T \leq C_T\sqrt{\varepsilon+\delta}. \end{aligned}$$

• *Moreover, there exist $x,k \in C([0,T];\mathbb{R}^d)$ and $h\in L^2(0,T;\mathbb{R}^d)$, such that*

$$\lim_{\varepsilon\to 0} k_\varepsilon(t) = k(t) = \int_0^t h(s)\,ds, \textit{for all } t\in[0,T],$$
$$\lim_{\varepsilon\to 0} \|x_\varepsilon - x\|_T = 0$$

and (x,k) is the unique solution of the variational inequality with oblique subgradients (15).

Corollary 14 *If $(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{t\geq 0})$ is a stochastic basis and M a $\mathcal{F}_t-$progressively measurable stochastic process such that $M_\cdot(\omega)\in C^1(\mathbb{R}_+;\mathbb{R}^d)$, $\mathbb{P}-a.s.\ \omega\in\Omega$, then, under the assumptions of Proposition 12, $\mathbb{P}-a.s.\ \omega\in\Omega$, the random generalized Skorohod problem with oblique subgradients:*

$$\begin{cases} X_t(\omega) + \displaystyle\int_0^t H(X_t(\omega))\,dK_t(\omega) = x_0 + \int_0^t f(s,X_s(\omega))\,ds + M_t(\omega), \quad t\geq 0,\\ dK_t(\omega) \in \partial\varphi(X_t(\omega))(dt) \end{cases}$$

admits a unique solution $(X_\cdot(\omega), K_\cdot(\omega))$. Moreover X and K are $\mathcal{F}_t-$progressively measurable stochastic processes.

2.2 Stochastic variational inequalities with oblique reflection

2.2.1 Formulation of the problem and main assumptions

In this section we introduce the Stochastic Variational Inequalities (for short, SVIs) with oblique subgradient and the definition of their strong and weak solutions. The theorem of existence and uniqueness results are given in the next section.

Let $(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{t\geq 0})$ be a stochastic basis and $\{B_t : t\geq 0\}$ a $\mathbb{R}^k-$valued Brownian motion. We will solve the following SVI with oblique reflection

$$(19)\qquad \begin{cases} X_t + \displaystyle\int_0^t H(X_t)\,dK_t = x_0 + \int_0^t f(s,X_s)\,ds + \int_0^t g(s,X_s)\,dB_s, \quad t\geq 0,\\ dK_t \in \partial\varphi(X_t)(dt), \end{cases}$$

where $x_0 \in \mathbb{R}^d$ and

$$(20)\qquad \begin{array}{ll} (i) & (t,x) \longmapsto f(t,x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d \text{ and } (t,x) \longmapsto g(t,x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d\times k} \text{ are} \\ & \quad \text{Carathéodory functions (i.e. measurable w.r. to } t \text{ and continuous w.r. to } x\text{),} \\ (ii) & \displaystyle\int_0^T (f^{\#}(t))^2 dt + \int_0^T (g^{\#}(t))^4 dt < \infty, \end{array}$$

with $f^{\#}(t) \overset{def}{=} \sup_{x\in Dom(\varphi)} |f(t,x)|$ and $g^{\#}(t) \overset{def}{=} \sup_{x\in Dom(\varphi)} |g(t,x)|$. We also add Lipschitz continuity conditions:

$$(21)\qquad \begin{array}{ll} & \exists\, \mu \in L^1_{loc}(\mathbb{R}_+),\ \ \exists\, \ell \in L^2_{loc}(\mathbb{R}_+) \text{ s.t. } \forall\, x,y \in \mathbb{R}^d, \quad a.e.\ t \geq 0, \\ (i) & |f(t,x) - f(t,y)| \leq \mu(t)\,|x-y|, \\ (ii) & |g(t,x) - g(t,y)| \leq \ell(t)\,|x-y|. \end{array}$$

Definition 15 *(I) Given a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ and a $\mathbb{R}^k-$valued $\mathcal{F}_t-$Brownian motion $\{B_t : t \geq 0\}$, a pair $(X,K) : \Omega\times[0,\infty[\to \mathbb{R}^d\times\mathbb{R}^d$ of continuous $\mathcal{F}_t-$progressively measurable stochastic processes is a strong solution of the SDE (19) if, $\mathbb{P}-a.s.\ \omega \in \Omega$:*

$$(22)\qquad \left\{ \begin{array}{ll} i) & X_t \in \overline{Dom(\varphi)},\ \ \forall\, t \geq 0,\ \varphi(X_\cdot) \in L^1_{loc}(\mathbb{R}_+), \\ ii) & K_\cdot \in BV_{loc}\left([0,\infty[\,;\mathbb{R}^d\right), \qquad K_0 = 0, \\ iii) & X_t + \displaystyle\int_0^t H(X_s)\,dK_s = x_0 + \int_0^t f(s,X_s)\,ds + \int_0^t g(s,X_s)\,dB_s,\ \forall\, t \geq 0, \\ iv) & \forall\, 0 \leq s \leq t,\ \forall y : \mathbb{R}_+ \to \mathbb{R}^d \text{ continuous} : \\ & \qquad \displaystyle\int_s^t \langle y(r) - X_r, dK_r\rangle + \int_s^t \varphi(X_r)\,dr \leq \int_s^t \varphi(y(r))\,dr. \end{array} \right.$$

That is

$$(X_\cdot(\omega), K_\cdot(\omega)) \in \mathcal{SP}(H\partial\varphi; x_0, M_\cdot(\omega)), \quad \mathbb{P}-a.s.\ \omega \in \Omega,$$

with

$$M_t = \int_0^t f(s,X_s)\,ds + \int_0^t g(s,X_s)\,dB_s\,.$$

(II) If there exists a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)_{t\geq 0}$, a $\mathbb{R}^k-$valued $\mathcal{F}_t-$Brownian motion $\{B_t : t \geq 0\}$ and a pair $(X_\cdot, K_\cdot) : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d \times \mathbb{R}^d$ of $\mathcal{F}_t-$progressively measurable continuous stochastic processes such that

$$(X_\cdot(\omega), K_\cdot(\omega)) \in \mathcal{SP}(H\partial\varphi; x_0, M_\cdot(\omega)), \quad \mathbb{P}-a.s.\ \omega \in \Omega,$$

then the collection $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, B_t, X_t, K_t)_{t\geq 0}$ is called a weak solution of the SVI (19).
(In both cases (I) and (II) we will say that (X,K) is a solution of the oblique reflected SVI (19).)

We introduce the spaces that will appear furthermore in our study. Denote by $S^p_d[0,T]$, $p \geq 0$, the space of progressively measurable continuous stochastic processes $X : \Omega \times [0,T] \to \mathbb{R}^d$, such that

$$\|X\|_{S^p_d} = \left\{ \begin{array}{ll} \left(\mathbb{E}\,\|X\|^p_T\right)^{\frac{1}{p}\wedge 1} < \infty, & \text{if } p > 0, \\ \mathbb{E}\left[1 \wedge \|X\|_T\right], & \text{if } p = 0, \end{array} \right.$$

where $\|X\|_T = \sup_{t\in[0,T]} |X_t|$. The space $(S_d^p[0,T], \|\cdot\|_{S_d^p})$, $p \geq 1$, is a Banach space and $S_d^p[0,T]$, $0 \leq p < 1$, is a complete metric space with the metric $\rho(Z_1, Z_2) = \|Z_1 - Z_2\|_{S_d^p}$ (when $p = 0$ the metric convergence coincides with the probability convergence).

Denote by $\Lambda_{d\times k}^p(0,T)$, $p \in [0,\infty)$, the space of progressively measurable stochastic processes $Z : \Omega \times (0,T) \to \mathbb{R}^{d\times k}$ such that

$$\|Z\|_{\Lambda^p} = \begin{cases} \left[\mathbb{E}\left(\int_0^T \|Z_s\|^2 ds\right)^{\frac{p}{2}}\right]^{\frac{1}{p}\wedge 1}, & \text{if } p > 0, \\ \mathbb{E}\left[1 \wedge \left(\int_0^T \|Z_s\|^2 ds\right)^{\frac{1}{2}}\right], & \text{if } p = 0. \end{cases}$$

The space $(\Lambda_{d\times k}^p(0,T), \|\cdot\|_{\Lambda^p})$, $p \geq 1$, is a Banach space and $\Lambda_{d\times k}^p(0,T)$, $0 \leq p < 1$, is a complete metric space with the metric $\rho(Z_1, Z_2) = \|Z_1 - Z_2\|_{\Lambda^p}$.

2.2.2 Existence and uniqueness of the solution

In this section we give an existence and uniqueness result for the solution of the stochastic variational inequality with oblique subgradients introduced before. Theorem 16 deals with the existence of a weak solution in the sense of Definition 15, while Theorem 17 gives the uniqueness of a strong solution.

Theorem 16 *Let the assumptions (7), (5), (9) and (20) be satisfied. Then the SVI (19) has at least one weak solution* $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, B_t, X_t, K_t)_{t\geq 0}$.

Proof. The proof is divided in three main steps. We construct a sequence of approximating equations, whose unique sequence of solutions is tight in $C([0,T]; \mathbb{R}^{2d+1})$, which permits us to make use of the Prohorov and Skorokod theorems. Finally, we pass to the limit in order to obtain a weak solution for the SVI (19). For the weak existence we will work in the framework of the Meyer-Zheng topology. ■

Theorem 17 *If the assumptions (7), (5), (9), (20) and (21) are satisfied, then the SVI (19) has a unique strong solution* $(X, K) \in S_d^0 \times S_d^0$.

Proof. For the proof of the above result it is sufficient to prove the *pathwise uniqueness*, since *the existence of a weak solution* and *the pathwise uniqueness* implies the existence of a strong solution. ■

2.2.3 A particular case of a perturbed convex set

We consider, as a particular case, the situation when we have, as the subdifferential operator, the subdifferential of the convex indicator of a time perturbed set and we prove that we can reduce the classical problem of SVIs with normal reflection to a SVIs with oblique subgradients, but with a fixed convex set which provides the multivalued operator.

Example 18 *Consider the stochastic variational inequality*

(23) $$\begin{cases} dX_t + \partial I_{L(t)E}(X_t)(dt) \ni f(t, X_t) dt + g(t, X_t) dB_t, & t \geq 0, \\ X_0 = x_0, \end{cases}$$

where $E \subset \mathbb{R}^d$ is a nonempty closed convex set, $L : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function or $L = (l_{i,j})_{d\times d} \in C_b^2\left(\mathbb{R}_+; \mathbb{R}^{d\times d}\right)$ is a symmetric matrix satisfying the conditions imposed to H in the previous results. We denote $Y_t = [H(t)]^{-1} X_t$ and we obtain, by the regularity of L, that Y must be a solution of the following SVI

(24) $$\begin{cases} dY_t + ([L(t)]^{-1})^2 \partial I_E(Y_t)(dt) \ni \tilde{f}(t, Y_t)dt + \tilde{g}(t, Y_t) dB_t, & t \geq 0, \\ Y_0 = [L(0)]^{-1} x_0, \end{cases}$$

where $\tilde{f}(t, Y_t) = [L(t)]^{-1}(f(t, L(t) Y_t) - L'(t) Y_t)$ and $\tilde{g}(t, Y_t) = [L(t)]^{-1} g(t, L(t) Y_t)$. For the transformed SVI (24) we can apply previous results in order to obtain the existence and the uniqueness of a strong solution.

3 Backward stochastic variational inequalities with oblique subgradients

This chapter is dedicated to the study of backward stochastic variational inequalities with oblique reflection, following the same framework constructed in the previous chapter. In fact, we consider two different problems, which differ through the form of the matrix H. Starting at a certain point, this difference will lead the proof of the existence of a solution on two different paths. First, we present the hypothesis imposed on the coefficients and we formulate the main results of our study on this topic.

3.1 Setting the problem

Let $T > 0$ be fixed and consider the backward stochastic variational inequality with oblique reflection (for short, $BSVI(H(t,y), \varphi, F)$, $BSVI(H(t), \varphi, F)$ or, respectively, $BSVI(H(y), \varphi, F)$ if the matrix H depends only on the time or, respectively, on the state of the system), $\mathbb{P} - a.s.$,

(25) $$\begin{cases} Y_t + \int_t^T H(s, Y_s) dK_s = \eta + \int_t^T F(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s, & t \in [0, T], \\ dK_s \in \partial\varphi(Y_s)(ds), \end{cases}$$

where

(H_1) $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ is a stochastic basis and $\{B_t : t \geq 0\}$ is a $\mathbb{R}^k-$valued Brownian motion. Moreover, the native filtration on Ω is given by $\mathcal{F}_t = \mathcal{F}_t^B = \sigma(\{B_s : 0 \leq s \leq t\}) \vee \mathcal{N}$.

(H_2) $H(\cdot,\cdot,y):\Omega\times\mathbb{R}_+\to\mathbb{R}^{d\times d}$ is progressively measurable for every $y\in\mathbb{R}^d$; there exists $\Lambda,b>0$ such that $\mathbb{P}-a.s.\ \omega\in\Omega$, $H=(h_{i,j})_{d\times d}\in C^{1,2}\left(\mathbb{R}_+\times\mathbb{R}^d;\mathbb{R}^{d\times d}\right)$ and, for all $t\in[0,T]$ and $y,\tilde{y}\in\mathbb{R}^d$, $\mathbb{P}-a.s.\ \omega\in\Omega$,

$$(26)\qquad \begin{cases} (i) & h_{i,j}(t,y)=h_{j,i}(t,y), \quad \forall i,j\in\overline{1,d},\\ (ii) & \langle H(t,y)u,u\rangle\geq a|u|^2, \quad \forall u\in\mathbb{R}^d \text{ (for some } a\geq 1),\\ (iii) & |H(t,\tilde{y})-H(t,y)|+|[H(t,\tilde{y})]^{-1}-[H(t,y)]^{-1}|\leq\Lambda|\tilde{y}-y|,\\ (iv) & |H(t,y)|+|[H(t,y)]^{-1}|\ \leq b,\end{cases}$$

where $|H(x)|\overset{def}{=}\left(\sum_{i,j=1}^d|h_{i,j}(x)|^2\right)^{1/2}$.

(H_3) the function

$$\varphi:\mathbb{R}^d\to(-\infty,+\infty] \text{ is a proper lower semicontinuous convex function.}$$

The generator function $F(\cdot,\cdot,y,z):\Omega\times[0,T]\to\mathbb{R}^d$ is progressively measurable for every $(y,z)\in\mathbb{R}^d\times\mathbb{R}^{d\times k}$ and there exist $L,\ell,\rho\in L^2(0,T;\mathbb{R}_+)$ such that

$$(H_4)\qquad \begin{cases} (i)\ \ \textit{Lipschitz conditions: for all } y,y'\in\mathbb{R}^d,\ z,z'\in\mathbb{R}^{d\times k},\ d\mathbb{P}\otimes dt-a.e.:\\ \qquad |F(t,y',z)-F(t,y,z)|\leq L(t)|y'-y|,\\ \qquad |F(t,y,z')-F(t,y,z)|\leq\ell(t)|z'-z|;\\ (ii)\ \ \textit{Boundedness condition:}\\ \qquad |F(t,0,0)|\leq\rho(t), \quad d\mathbb{P}\otimes dt-a.e..\end{cases}$$

We introduce now the notion of solution for Eq.(25). We will study two types of solution, given by the following Definitions. For the case $H(t,y)\equiv H(t)$ we obtain the existence of a strong solution while, for the general case $H(t,y)$ we obtain a weak solution for Eq.(25).

Definition 19 *Given $(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{t\geq0})$ a fixed stochastic basis and $\{B_t:t\geq0\}$ a $\mathbb{R}^k-$valued Brownian motion, we state that a triplet (Y,Z,K) is a strong solution of the $BSVI(H(t),\varphi,F)$ if $(Y,Z,K):\Omega\times[0,T]\to\mathbb{R}^d\times\mathbb{R}^{d\times k}\times\mathbb{R}^d$ are progressively measurable continuous stochastic processes and $\mathbb{P}-a.s.\ \omega\in\Omega$,*

$$\begin{cases} Y_t+\int_t^T H(s)\,dK_s=\eta+\int_t^T F(s,Y_s,Z_s)\,ds-\int_t^T Z_s dB_s, \quad \forall t\in[0,T],\\ dK_s\in\partial\varphi(Y_s)(ds).\end{cases}$$

Consider now the case when the matrix H depends on the state of the system. We can reconsider the backward stochastic variational inequality with oblique reflection in the following manner, $\mathbb{P}-a.s.\ \omega\in\Omega$,

$$(27)\qquad \begin{cases} Y_t+\int_t^T H(s,Y_s)\,dK_s=\eta+\int_t^T F(s,Y_s,Z_s)\,ds-(M_T-M_t), \quad \forall t\in[0,T],\\ dK_s\in\partial\varphi(Y_s)(ds),\end{cases}$$

where M is a continuous martingale (possible with respect to its natural filtration if not any other filtration available). If

$$H(\omega,t,y)\equiv H(t,y) \quad \text{and} \quad F(\omega,t,y,z)\equiv F(t,y,z)$$

we introduce the notion of weak solution of the equation.

Definition 20 *If there exists a probability space $(\Omega,\mathcal{F},\mathbb{P})$ and a triplet $(Y,M,K):\Omega\times[0,T]\rightarrow(\mathbb{R}^d)^3$ such that*

(a) *M is a continuous martingale with respect to the filtration given, for $\forall t\in[0,T]$, by $\mathcal{F}_t\overset{def}{=}\mathcal{F}_t^{Y,M}=\sigma(\{Y_s,M_s:0\leq s\leq t\})\vee\mathcal{N}$,*

(b) *Y,K are càdlàg stochastic processes, adapted to $\{\mathcal{F}_t\}_{t\geq0}$,*

(c) *relation (27) is verified for every $t\in[0,T]$, $\mathbb{P}-a.s.\ \omega\in\Omega$,*

the collection $(\Omega,\mathcal{F},\mathbb{P},\mathcal{F}_t,Y_t,M_t,K_t)_{t\in[0,T]}$ is called a weak solution of the $BSVI\,(H(y),\varphi,F)$.

In both cases given by Definition 19 or Definition 20 we will say that (Y,Z,K) or (Y,M,K) is a solution of the considered oblique reflected backward stochastic variational inequality.

3.2 Main results - the common core

3.2.1 Formulation of the existence and uniqueness theorems

Denote

$$\nu_t=\int_0^t L(s)\left[\mathbb{E}^{\mathcal{F}_s}|\eta|^p\right]^{1/p} \quad \text{and} \quad \theta=\sup_{t\in[0,T]}\left(\mathbb{E}^{\mathcal{F}_t}|\eta|^p\right)^{1/p}.$$

Theorem 21 *Let $p>1$ and the assumptions (H_1-H_4) be satisfied. If*

$$\mathbb{E}e^{\delta\theta}+\mathbb{E}\left|\varphi(\eta)\right|<\infty \tag{28}$$

for all $\delta>0$ then the $BSVI\,(H(t),\varphi,F)$ admits a unique strong solution $(Y,Z,K)\in S_d^0[0,T]\times\Lambda_{d\times k}^0(0,T)\times S_d^0[0,T]$ such that, for all $\delta>0$,

$$\mathbb{E}\sup_{s\in[0,T]}e^{\delta p\nu_s}|Y_s|^p+\mathbb{E}\left(\int_0^T e^{2\delta\nu_s}|Z_s|^2ds\right)^{p/2}<\infty. \tag{29}$$

Moreover, there exists a positive constant, independent of the terminal time T, $C=C(a,b,\Lambda)$ such that, $\mathbb{P}-a.s.\ \omega\in\Omega$,

$$|Y_t|\leq C\left(1+\left[\mathbb{E}^{\mathcal{F}_t}|\eta|^p\right]^{1/p}\right), \quad \text{for all } t\in[0,T]$$

and the process K is absolutely continuous and, therefore, it can be represented as

$$K_t=\int_0^t U_s ds,$$

where

$$\mathbb{E}\int_0^T|U_t|^2dt+\mathbb{E}\int_0^T|Z_t|^2dt\leq C\left(\mathbb{E}|\eta|^2+\mathbb{E}\left|\varphi(\eta)\right|+\mathbb{E}\int_0^T|F(t,0,0)|^2dt\right).$$

Theorem 22 *Let the assumptions $(H_2 - H_4)$ be satisfied. Then the $BSVI\left(H\left(t,y\right),\varphi,F\right)$ (25) admits a unique weak solution $(\Omega,\mathcal{F},\mathbb{P},\mathcal{F}_t,B_t,Y_t,M_t,K_t)_{t\in[0,T]}$.*

The proofs of the above results are detailed in the next sections. Section 3.2.2 deals with a sequence of approximating equations and apriori estimates of their solutions. The estimates will be valid for both cases covered by Theorem 21 and Theorem 22. After this, the proof is split between Section 3.3 and Section 3.4, each one being dedicated to the particularities brought by Theorem 21 and Theorem 22.

3.2.2 Approximating problems and apriori estimates

We start simultaneously the profs of Theorem 21 and Theorem 22 by obtaining some apriori estimates for the solutions of the approximating equations. Let $0<\varepsilon\leq 1$. Consider the approximating BSDE

$$(30)\qquad Y_t^\varepsilon+\int_t^T H\left(s,Y_s^\varepsilon\right)\nabla\varphi_\varepsilon\left(Y_s^\varepsilon\right)ds=\eta+\int_t^T F\left(s,Y_s^\varepsilon,Z_s^\varepsilon\right)ds-\int_t^T Z_s^\varepsilon dB_s,\quad \forall t\in[0,T].$$

and we obtain the estimates given by the following result.

Lemma 23 *Consider the approximating BSDE (30), with its solution $(Y^\varepsilon,Z^\varepsilon)$ and denote $U^\varepsilon=\nabla\varphi_\varepsilon(Y^\varepsilon)$. There exists a positive constant $C=C(a,b,\Lambda,l,L(\cdot))$, independent of ε, such that*

$$(31)\qquad \mathbb{E}\sup_{s\in[0,T]}|Y_s^\varepsilon|^2+\mathbb{E}\int_0^T(|U_r^\varepsilon|^2+|Z_r^\varepsilon|^2)dr\leq C\left[\mathbb{E}\left|\eta\right|^2+\mathbb{E}\varphi\left(\eta\right)+\mathbb{E}\int_0^T\left|F\left(r,0,0\right)\right|^2dr\right].$$

3.3 Strong existence and uniqueness for $H(t,y)\equiv H_t$

Using the results given by Lemma 23 we continue the proof of the existence of a strong solution. Under the assumptions of Step 3 (Section 3.2.2) we prove that $\{Y^\varepsilon:0<\varepsilon\leq 1\}$ is a Cauchy sequence. With standard arguments, passing to the limit in the approximating equation (30) we infer that

$$Y_t+\int_t^T H_sU_sds=\eta+\int_t^T F(s,Y_s,Z_s)ds-\int_t^T Z_sdB_s,\quad \forall t\in[0,T].$$

Moreover, since $\nabla\varphi_\varepsilon(x)\in\partial\varphi(J_\varepsilon x)$ we have, on the subsequence ε_n,

$$\mathbb{E}\int_0^T\left\langle\nabla\varphi_{\varepsilon_n}(Y_t^{\varepsilon_n}),v_t-Y_t^{\varepsilon_n}\right\rangle dt+\mathbb{E}\int_0^T\varphi(J_{\varepsilon_n}(Y_t^{\varepsilon_n}))dt\leq\mathbb{E}\int_0^T\varphi(v_t)dt,$$

for every progressively measurable continuous stochastic process v. Hence $U_s\in\partial\varphi(Y_s)$ for every $s\in[0,T]$, $\mathbb{P}-a.s.\ \omega\in\Omega$ and we can conclude that the triplet (Y,Z,K) is a strong solution of the $BSVI\left(H\left(t\right),\varphi,F\right)$. The uniqueness of the solution is also proved.

3.4 Weak existence for $H(t,y)$

Allowing the dependence on Y we will situate ourselves in a Markovian framework and one will use tightness criteria in order to prove the existence of a weak solution for the problem $BSVI\left(H\left(t,y\right),\varphi,F\right)$. First let $b:[0,T]\times\mathbb{R}^k\to\mathbb{R}^k$, $\sigma:[0,T]\times\mathbb{R}^k\to\mathbb{R}^{k\times k}$ be two measurable functions satisfying the classical conditions, which imply the existence of a non-exploding solution for the following SDE

$$X_s^{t,x}=x+\int_t^s b(r,X_r^{t,x})dr+\int_t^s\sigma(r,X_r^{t,x})dB_r,\quad t\le s\le T. \tag{32}$$

According to Friedmann [21] it follows that, for every $(t,x)\in[0,T]\times\mathbb{R}^k$, the equation (32) admits a unique solution $X^{t,x}$. Moreover, for $p\ge 1$, there exists a positive constant $C_{p,T}$ such that

$$\begin{cases}\mathbb{E}\sup_{s\in[0,T]}|X_s^{t,x}|^p\le C_{p,T}(1+|x|^p)\quad\text{and}\\ \mathbb{E}\sup_{s\in[0,T]}|X_s^{t,x}-X_s^{t',x'}|^p\le C_{p,T}(1+|x|^p)(|t-t'|^{p/2}+|x-x'|^p),\end{cases} \tag{33}$$

for all $x,x'\in\mathbb{R}^k$ and $t,t'\in[0,T]$.

Let consider the continuous generator function $F:[0,T]\times\mathbb{R}^k\times\mathbb{R}^d\to\mathbb{R}^d$ and assume there exist $L\in L^2\left(0,T;\mathbb{R}_+\right)$ such that, for all $t\in[0,T]$ and $x\in\mathbb{R}^k$,

$$|F(t,x,y')-F(t,x,y)|\le L\left(t\right)|y'-y|,\quad \textit{for all } y,y'\in\mathbb{R}^d, \tag{H_4'}$$

Given a continuous function $g:\mathbb{R}^k\to\mathbb{R}^d$, satisfying a sublinear growth condition, consider now the $BSVI\left(H\left(t,y\right),\varphi,F\right)$

$$\begin{cases}Y_s^{t,x}+\displaystyle\int_s^T H(r,Y_r^{t,x})dK_r^{t,x}=g(X_T^{t,x})+\int_s^T F(r,X_r^{t,x},Y_r^{t,x})dr-\int_s^T Z_r^{t,x}dB_r,\quad t\le s\le T,\\ dK_r^{t,x}\in\partial\varphi(Y_r^{t,x})\left(dr\right),\quad\text{for every } r.\end{cases} \tag{34}$$

Assume also that all hypothesis given by (H_2) still hold for the deterministic matrix $H:[0,T]\times\mathbb{R}^d\to\mathbb{R}^{d\times d}$.

The solution will be constructed on the Skorohod space $\mathcal{D}([0,T];\mathbb{R}^m)$ of càdlàg functions $y:[0,T]\to\mathbb{R}^m$ (i.e. right continuous and with left-hand side limit). It can be shown (see Billingsley [7]) that, although $\mathcal{D}([0,T];\mathbb{R}^m)$ is not a complete space with respect to the Skorohod metric, there exists a topologically equivalent metric with respect to which it is complete and that the Skorohod space is a Polish space. The space of continuous functions $C([0,T];\mathbb{R}^m)$, equipped with the supremum norm topology is a subspace of $\mathcal{D}([0,T];\mathbb{R}^m)$; the Skorohod topology restricted to $C([0,T];\mathbb{R}^m)$ coincides with the uniform topology. We will use on $\mathcal{D}([0,T];\mathbb{R}^m)$ the Meyer-Zheng topology, which is the topology of convergence in measure, weaker than the Skorohod topology.

For any fixed $n\ge 1$ consider the following approximating equation, which is in fact BSDE (30) from Section 3, adapted to our new setup. We have, $\mathbb{P}-a.s.\ \omega\in\Omega$,

$$Y_t^n+\int_t^T H\left(s,Y_s^n\right)\nabla\varphi_{1/n}\left(Y_s^n\right)ds=g(X_T^{t,x})+\int_t^T F\left(s,X_s,Y_s^n\right)ds-\int_t^T Z_s^n dB_s,\quad\forall t\in[0,T]. \tag{35}$$

The estimations obtained in Section 3.2.2 apply also to the triplet $\left(Y^n, Z^n, \nabla\varphi_{1/n}\left(Y^n\right)\right)$. In the sequel we will employ the notations $M_t^n = \int_0^t Z_s^n dB_s$ and $K_t^n = \int_0^t \nabla\varphi_{1/n}\left(Y_s^n\right) ds$. Our goal is to prove the tightness of the sequence $\{Y^n, M^n\}_n$ with respect to the Meyer-Zheng topology. By passing to the limit in the approximating equations we will conclude the existence of a weak solution.

4 Variational inequalities with oblique subgradients on non-convex domains

In this chapter we study a deterministic differential inclusion of the type

$$dx(t) + H\left(x\left(t\right)\right)\partial^{-}\varphi\left(x\left(t\right)\right)\left(dt\right) \ni g\left(t\right),\ a.e.\ t\in\left[0,T\right],\ x\left(0\right)=x_0\in int\left(Dom\left(\varphi\right)\right),$$

where $\partial^{-}\varphi$ is the Fréchet subdifferential of φ introduced in Section 1.1.2. The effect produced by the matrix H consists in the generation of an oblique reflection, in the manner already discussed in Chapter 2 and Chapter 3. First, in Section 4.1, we present some additional properties of the (ρ,γ) −semiconvex functions and we will obtain fine estimations which will prepare the framework for the approximating setup. In Sections 4.2.1 and 4.2.2 we present the main achievements of this chapter.

4.1 Preliminaries, notations and hypothesis

Theorem 24 *Let $\varepsilon_0 > 0$ and $\varphi : \mathbb{R}^d \to (-\infty, +\infty]$ be a lower semicontinuous (l.s.c.) function such that $Dom\left(\varphi\right) \neq \emptyset$ and*

$$m_0 = \inf\left\{\frac{1}{2\varepsilon_0}\left|v\right|^2 + \varphi\left(v\right) : v\in\mathbb{R}^d\right\} > -\infty.$$

Let $u \in \mathbb{R}^d$ and $\varepsilon \in (0, \varepsilon_0)$ be fixed. Then for every $r > 0$ there exists $\hat{u} \in Dom\left(\varphi\right)$ and $\overline{u} \in B\left(u, r\right) = \left\{h\in\mathbb{R}^d : |h-u| < r\right\}$ such that

j)

$$\frac{1}{2\varepsilon}\left|\overline{u}-\hat{u}\right|^2 + \varphi\left(\hat{u}\right) = \inf_{v\in\mathbb{R}^d}\left\{\frac{1}{2\varepsilon}\left|\overline{u}-v\right|^2 + \varphi\left(v\right)\right\}$$

jj)

$$\frac{1}{2\varepsilon}\left|\overline{u}-\widehat{u}\right|^2 + \varphi\left(\widehat{u}\right) \leq \varphi\left(u\right). \tag{36}$$

Moreover, $\widehat{u} \in Dom\left(\partial^{-}\varphi\right)$ and $\frac{1}{\varepsilon}\left(\overline{u}-\widehat{u}\right) \in \partial^{-}\varphi\left(\widehat{u}\right)$.

Proposition 25 *Let $\varphi : \mathbb{R}^d \to (-\infty, +\infty]$ be a (ρ, γ) −semiconvex function. There exists $a \in \mathbb{R}$ such that*

$$\varphi(v) + a|v|^2 + a \geq 0, \quad \forall v \in \mathbb{R}^d. \tag{37}$$

Moreover, if we consider $0 < \varepsilon < \frac{1}{2a}$ arbitrary fixed and K is a closed subset of $\mathbb{R}^d$ then, for every $u \in K$ there exists $\hat{u} \in K \cap Dom(\varphi)$ such that

$$\frac{1}{2\varepsilon}|u - \hat{u}|^2 + \varphi(\hat{u}) = \inf_{v \in K \cap Dom(\varphi)} \left\{ \frac{1}{2\varepsilon}|u - v|^2 + \varphi(v) \right\} \stackrel{def}{=} \varphi_\varepsilon(u; K). \tag{38}$$

The following result presents a Lipschitz property for J_ε, which requires a boundedness conditions for A_ε. We will see later that, imposing some additional assumptions on φ that condition can be overcome.

Lemma 26 *Let $\varphi : \mathbb{R}^d \to (-\infty, +\infty]$ be a proper l.s.c. (ρ, γ) −semiconvex function, $S > 0$ and $0 < \varepsilon < \frac{1}{2(\rho + \gamma S)}$. Let $u, v \in \mathbb{R}^d$ and consider the pairs $(J_\varepsilon u, A_\varepsilon(u)), (J_\varepsilon v, A_\varepsilon(v)) \in \partial^- \varphi$. If $max\{|A_\varepsilon(u)|, |A_\varepsilon(v)|\} \leq S$ then*

$$|J_\varepsilon u - J_\varepsilon v| \leq \frac{1}{1 - 2\varepsilon(\rho + \gamma S)}|u - v|. \tag{39}$$

In the sequel we will impose some **additional hypothesis** on the (ρ, γ) −semiconvex function φ. We assume that

$$\begin{cases} i) & \varphi \text{ is locally bounded on } int(Dom(\varphi)), \\ ii) & \partial^-\varphi(u) \neq \emptyset, \quad \forall u \in int(Dom(\varphi)). \\ iii) & \overline{Dom(\varphi)} = \overline{int(Dom(\varphi))}. \end{cases} \tag{40}$$

Remark that, according to (40−iii)), we have the following equalities of sets: $int(Dom(\varphi)) = int(Dom(\partial^-\varphi))$ and $\overline{Dom(\varphi)} = \overline{Dom(\partial^-\varphi)}$. Also, $\nabla^-\varphi : int(Dom(\varphi)) \longrightarrow \mathbb{R}^d$ is bounded on bounded sets.
As consequence, the following important result takes place.

Proposition 27 *Let $\varphi : \mathbb{R}^d \to (-\infty, +\infty]$ be a (ρ, γ) −semiconvex function. If we consider $u_0 \in Dom(\varphi)$, $r_0, M_0 > 0$, satisfying $\varphi(u_0 + r_0 v) \leq M_0$, $\forall |v| \leq 1$, then there exist $\rho_0 > 0$ and $b \geq 0$ such that*

$$\rho_0|\hat{u}| \leq \langle \hat{u}, u - u_0 \rangle + b + b(1 + |\hat{u}|)|u - u_0|^2, \quad \forall (u, \hat{u}) \in \partial^-\varphi. \tag{41}$$

Moreover there exist $M \geq 0$ and $\delta_0 \in (0, r_0]$ such that

$$|\hat{u}| \leq M, \quad \forall u \in \overline{B(u_0, \delta_0)} \subset Dom(\varphi) \text{ and } \hat{u} \in \partial^-\varphi(u). \tag{42}$$

Theorem 28 *Let $\varphi : \mathbb{R}^d \to (-\infty, +\infty]$ be a proper l.s.c. (ρ, γ) −semiconvex function. Consider $u_0 \in Dom(\partial^-\varphi)$, $r_0, M_0 > 0$, satisfying $\varphi(u_0 + r_0 v) \le M_0$, $\forall |v| \le 1$ and let $\delta_0 \in (0, r_0]$ be such that*

$$\overline{B(u_0, \delta_0)} \subset int(Dom(\varphi))$$

and (42) takes place. Consider $u \in \overline{B(u_0, \delta_0)}$. For $\varepsilon > 0$ small enough and independent of u, the function $v \longrightarrow \varphi(v) + \frac{1}{2\varepsilon}|u - v|^2$ admits an unique minimizing point $u_\varepsilon (\overset{def}{=} J_\varepsilon u)$ in $\overline{B(u_0, \delta_0)}$. Moreover, if we denote

$$\varphi_\varepsilon(u) = \varphi_\varepsilon(u, \delta_0, u_0) \overset{def}{=} \inf_{v \in \overline{B(u_0,\delta_0)}} \left\{ \frac{1}{2\varepsilon}|u - v|^2 + \varphi(v) \right\},$$

then $\varphi_\varepsilon \in C^1(B(u_0, \delta_0))$ and $\nabla\varphi_\varepsilon(u) = A_\varepsilon(u) = \dfrac{1}{\varepsilon}(u - J_\varepsilon u) \in \partial^-\varphi(J_\varepsilon u)$.

4.2 Multivalued differential equations with oblique subgradients

4.2.1 Formulation of the problem

Consider the following Cauchy problem

$$\text{(43)} \qquad \begin{cases} x'(t) + H(x(t))\,\partial^-\varphi(x(t)) \ni g(t), & a.e.\ t \in [0, T] \\ x(0) = x_0 \in int(Dom(\varphi)), \end{cases}$$

where $g \in L^1([0, T]; \mathbb{R}^d)$, $\varphi : \mathbb{R}^d \to]-\infty, +\infty]$ is a proper, lower semicontinuous, (ρ, γ) −semiconvex function such that there exist the positive constants C_1, C_2 :

$$(H\varphi) : |\varphi(x) - \varphi(x)| \le C_1 + C_2|x - y|, \quad \forall x, y \in Dom(\varphi),$$

$H = (h_{i,j})_{d\times d} \in C_b^2(\mathbb{R}^d; \mathbb{R}^{d\times d})$ is a matrix, such that, for all $x \in \mathbb{R}^d$,

$$(H_H) : \begin{cases} i) & h_{i,j}(x) = h_{j,i}(x), \quad \forall\, i, j \in \overline{1, d}, \\ ii) & \dfrac{1}{c_H}|u|^2 \le \langle H(x)u, u\rangle \le c_H|u|^2, \quad \forall u \in \mathbb{R}^d \text{ (for some } c_H \ge 1). \end{cases}$$

Definition 29 *A pair (x, h) of functions $x, h : [0, T] \longrightarrow \mathbb{R}^d$ is a solution of equation (43) (and we will write $(x, h) \in \mathcal{GR}(H\partial^-\varphi; x_0, g)$) if*

$$\text{(44)} \qquad \begin{cases} i) & x(t) \in \overline{Dom(\varphi)}, \quad \forall t \ge 0,\ x \in C([0, T]; \mathbb{R}^d) \quad and\ h, \varphi(x) \in L^1(0, T; \mathbb{R}^d), \\ ii) & x(t) + \displaystyle\int_0^t H(x(r))\,h(r)\,dr = x_0 + \int_0^t g(r)\,dr, \quad \forall t \in [0, T], \\ iii) & h(t) \in \partial^-\varphi(x(t)), \quad a.e.\ t \in [0, T]. \end{cases}$$

Remark 30 *Condition (44−iii) is equivalent with*

$$\text{(45)} \qquad \int_s^t \langle y(r) - x(r), h(r)\rangle\,dr + \int_s^t \varphi(x(r))\,dr \le \int_s^t \varphi(y(r))\,dr + \int_s^t |y(r) - x(r)|^2 (\rho + \gamma|h(r)|)\,dr,$$

for all $0 \le s \le t \le T$ and $y \in C([0, T]; \mathbb{R}^d)$.

4.2.2 Existence and uniqueness result

Lemma 31 *If $h(t) \in \partial^{-}\varphi(x(t))$ and $\hat{h}(t) \in \partial^{-}\varphi(\hat{x}(t))$ then, for all $0 \le s \le t$ we have*

$$\int_s^t |x(r) - \hat{x}(r)|^2 \big(2\rho + \gamma|h(r)| + \gamma|\hat{h}(r)|\big) dr + \int_s^t \Big\langle x(r) - \hat{x}(r), h(r) - \hat{h}(r) \Big\rangle dr \ge 0. \tag{46}$$

Theorem 32 *Let the assumptions $(H\varphi)$ and (H_H) be satisfied. The generalized non-convex differential system with oblique subgradients (43) admits a unique solution (x, h). Moreover, if $(x, h) \in \mathcal{GR}(H\partial^{-}\varphi; x_0, g)$ and $(\hat{x}, \hat{h}) \in \mathcal{GR}(H\partial^{-}\varphi; x_0, \hat{g})$ then*

$$|x(t) - \hat{x}(t)| \le Ce^{CU(t)} \left[|x_0 - \hat{x}_0| + \int_0^t |g(r) - \hat{g}(r)| dr\right], \tag{47}$$

where $U(t) = \int_0^t |x(r)| dr + \int_0^t |\hat{x}(r)| dr + (1+\gamma)\int_0^t |h(r)| dr + (1+\gamma)\int_0^t |\hat{h}(r)| dr + 2\rho t$ and C is a constant depending only on L_H and c_H.

5 A Skorohod problem driven by the Clarke subdifferential; applications to SDEs

5.1 Preliminaries and hypothesis

We recall the following definitions:

Definition 33 *A multivalued operator G from a topological space $\mathbb{U}$ to a topological space $\mathbb{V}$ is said to be upper semicontinuous if:*
- for all $u \in \mathbb{U}$, Gu is a compact subset of $\mathbb{V}$;
- for all $u \in \mathbb{U}$, for every $V \in \mathcal{V}_{\mathbb{V}}(Gu)$, there is a $U \in \mathcal{V}_{\mathbb{U}}(u)$ such that for all $z \in U$ we have $Gz \subset V$, where $\mathcal{V}_{\mathbb{X}}(x)$ denote the set of neighborhoods of x in $\mathbb{X}$.

Definition 34 *A multivalued operator $\Theta : \mathbb{U} \rightrightarrows \mathbb{V}$ is said to be measurable if $\Theta^{-}V \overset{def}{=} \{u \in \mathbb{U} : \Theta(u) \cap V \neq \emptyset\}$ is measurable, for every closed $V \subset \mathbb{V}$.*

Consider the multivalued equation

$$\begin{cases} dx(t) + \partial\varphi(x(t))\,dt + \partial_C\phi(x(t))\,dt + G(t, x(t))dt \ni dm(t), & t \in [0, T], \\ x(0) = x_0 \in \overline{D(\partial_C\phi)} \cap \overline{D(\varphi)}, \end{cases} \tag{48}$$

where $\partial_C\phi$ is the Clarke subdifferential operator associated to the function ϕ, $\partial\varphi$ is the classical subdifferential operator associated to a convex, l.s.c function φ, $G(t, x)$ is a time dependent multivalued operator on $\mathbb{R}^d$ and $m : [0, T] \to \mathbb{R}^d$ is a continuous function.

We suppose that there exist the positive constants c and M such that

$$(49)\qquad \begin{array}{ll} i) & |\partial_C \phi(x)| \le c(1+|x|), \quad \text{for all } x\in\mathbb{R}^d; \\ ii) & \forall x_1,x_2\in\mathbb{R}^d,\ \forall y_1\in\partial_C\phi(x_1),\ \forall y_2\in\partial_C\phi(x_2),\ \langle x_1-x_2,y_1-y_2\rangle \ge -M|x_1-x_2|^2 \end{array}$$

and

$$(50)\qquad \left\{ \begin{array}{ll} i) & \varphi:\mathbb{R}^d\to\,]-\infty,+\infty] \text{ is a proper convex l.s.c. function} \\ ii) & int\left(Dom\left(\varphi\right)\right)\neq\emptyset. \end{array}\right.$$

Definition 35 *A multivalued mapping $G:[0,T]\times\overline{D(\varphi)}\rightrightarrows\mathbb{R}^d$ is said to satisfy conditions (R), if the following conditions hold:*
(a) for almost all t in $[0,T]$, the mapping $x\to G(t,x)$ is multivalued u.s.c. from $\overline{D(\varphi)}$ to $\mathbb{R}^d$, with convex values.
(b) for all $x\in\overline{D(\varphi)}$, the mapping $t\to G(t,x)$ is multivalued measurable from $[0,T]$ to $\mathbb{R}^d$ (i.e., for all $\chi\in\mathbb{R}^d$ and all $x\in\overline{D(\varphi)}$, the function $\upsilon_{x,\chi}:t\to\sup\{\langle y,\chi\rangle : y\in G(t,x)\}$ is measurable on $[0,T]$).
(c) there exist two functions γ and δ from $L^p_{loc}(0,T)$, with $1\le p\le\infty$, such that for almost all $t\in[0,T]$ and all $x\in\overline{D(\varphi)}$,

$$\sup_{y\in G(t,x)}|y|\le\gamma(t)|x|+\delta(t).$$

We now suppose that the multifunction $G:I\times\overline{D(\varphi)}\to\mathbb{R}^d$ satisfies both the condition (R) and the following one-sided Lipschitz continuous condition: there exists $\eta\in L^1([0,T],\mathbb{R}^+)$ such that, for every $x,y\in\mathbb{R}^d$, $t\in[0,T]$ and $u\in G(t,x),\ v\in G(t,y)$ we have

$$(51)\qquad \langle v-u,y-x\rangle\ge-\eta(t)|x-y|^2.$$

Definition 36 *Given two functions $x,l:[0,T]\to\mathbb{R}^d$ and a locally Lipschitz function f, we say that $dl(t)\in\partial_C f(x(t))(dt)$, if for all $T\ge0$,*

$$(52)\qquad \begin{array}{ll} (a) & x,l:[0,T]\to\mathbb{R}^d \text{ are continuous,} \\ (b) & l\in BV_{loc}\left(\mathbb{R}_+;\mathbb{R}^d\right),\quad l(0)=0, \\ (c) & \displaystyle\int_s^t\langle v(r),dl(r)\rangle\le\int_s^t f^o(x(r),v(r))\,dr, \\ & \qquad\text{for all } 0\le s\le t\le T \quad\text{and } v\in C\left([0,T];\mathbb{R}^d\right). \end{array}$$

Definition 37 *A triplet of function (x,j,l) is called a solution of the evolution inclusion with Clarke subdifferential (48) if $x,j,l:[0,T]\to\mathbb{R}^d$ are continuous, and*

$$\left\{ \begin{array}{ll} i) & x(t)\in\overline{D(\varphi)},\forall t\ge0, \\ ii) & j,l\in BV\left([0,T];\mathbb{R}^d\right),\quad \text{with } j(0)=l(0)=0, \\ iii) & x(t)+j(t)+l(t)+\displaystyle\int_0^t\beta(s)\,ds=x_0+m(t), \\ iv) & dj(t)\in\partial\varphi(x(t))(dt),\quad dl(t)\in\partial_C\phi(x(t))(dt)\quad\text{and}\quad\beta(t)\in G(t,x(t)). \end{array}\right.$$

5.2 Existence and uniqueness results

In order to prove the existence and uniqueness of the solution for the multivalued equation (48), we first prove the following auxiliary theorem.

Theorem 38 *Let G be a time dependent multivalued operator on $\mathbb{R}^d$ satisfying conditions (R), φ satisfying (50) and $m \in \mathcal{M}$ ($\mathcal{M}$ is a bounded and equicontinuous subset of $C\left([0,T];\mathbb{R}^d\right)$). Then the following variational inequality admits a solution (x,j) on $[0,T]$:*

$$dx(t) + \partial\varphi(x(t))\,dt + G(t,x(t))\,dt \ni dm(t), \quad x(0) = x_0. \tag{53}$$

More precisely:
(1) *there exists a measurable selection $\beta : [0,T] \to \mathbb{R}^d$ such that $\beta(t) \in G(t,x(t))$ almost everywhere in $[0,T]$;*
(2) *(x,j) is a solution of $dx(t) + \partial\varphi(x(t))\,dt \ni dm(t) - \beta(t)\,dt$, $x(0) = x_0$.*

The main result of this section is given below.

Theorem 39 *If ϕ satisfies the assumptions (49), φ satisfies (50) and G verifies condition (R) and (51), there exists at least one solution (x,j,l) for the multivalued equation (48). Moreover, if $\mathcal{M}$ is a bounded and equicontinuous subset of $C\left([0,T];\mathbb{R}^d\right)$, then there exists a positive constant $C_{0,\mathcal{M}}$ such that:*
1. If $m \in \mathcal{M}$ and (x,j,l) is a solution of (48), then

$$\|x\|_T^2 + \updownarrow j \updownarrow_T + \updownarrow l \updownarrow_T \leq C_{0,\mathcal{M}}(1 + |x_0|^2). \tag{54}$$

2. If $(x_1,j_1,l_1),(x_2,j_2,l_2)$ are two solutions of (48) corresponding to the singular inputs $m_1,\ m_2$ and, respectively to the initial data $x_{1,0}$, $x_{2,0}$, then

$$\|x_1 - x_2\| \leq C_{0,\mathcal{M}}\left(1 + |x_{1,0}| + |x_{2,0}|\right)\left(|x_{1,0} - x_{2,0}| + \|m_1 - m_2\|_T^{1/2}\right). \tag{55}$$

In particular, if $x_{1,0} = x_{2,0}$ and $m_1 = m_2$, then we obtain the uniqueness of the solution.
3. For every solution (x,j,l) with initial data $x_0 \in \overline{D(\varphi)}$ and singular input $dm(t)$, the mapping $(x_0,m) \to x : \overline{D(\varphi)} \times C\left([0,T];\mathbb{R}^d\right) \to C([0,T];\overline{D(\varphi)})$ is continuous.

5.3 Stochastic variational inequalities with Clarke subdifferential

We consider the stochastic variational inequality

$$\begin{cases} dX_t + \partial\varphi(X_t)\,dt + \partial_C\phi(X_t)\,dt + G(t,X_t)dt \ni Q(t,X_t)dB_t, & t \in [0,T], \\ X_0 = \xi \in \overline{D(\varphi)}. \end{cases} \tag{56}$$

where ϕ satisfies the assumptions (49) and G satisfies condition (51). We assume also that $Q(.,.,x) : \Omega \times [0,T] \to \mathbb{R}^{d\times k}$ is a Carathéodory function and satisfies the following conditions: there exists $\ell \in L^2_{loc}(\mathbb{R}_+)$ such that

$$\begin{cases} |Q(t,x) - Q(t,y)| \leq \ell(t)\,|x-y|, \quad \forall x,y \in \mathbb{R}^d \quad \text{(Lipschitz condition)}, \\ \displaystyle\int_0^T |Q(t,0)|^2\,dt < \infty, \quad \mathbb{P}-a.s.\ \omega \in \Omega \quad \text{(boundedness condition)}. \end{cases} \tag{57}$$

We introduce below the notion of solution for the stochastic variational inequality (56).

Definition 40 *A quadruple* (X, J, K, β) *of* $\mathbb{R}^d-$*valued stochastic processes is a solution of (56) if the following conditions are satisfied,* $\mathbb{P}-a.s.\ \omega \in \Omega$, *for evert* $T > 0$:

$$\begin{cases} d_1) \quad X, J, K, \beta \in S_d^0[0,T], \quad J_0 = K_0 = 0, \\ d_2) \quad X_t \in \overline{D(\varphi)}, \text{ a.e.}, \quad \updownarrow j \updownarrow_T < \infty \text{ and } \updownarrow K \updownarrow_T < \infty, \\ d_3) \quad \beta_t \in G(t, X_t), \text{ a.e.}, \\ d_4) \quad X_t + J_t + K_t + \beta_t = \xi + \int_0^t Q(s, X_s)\, dB_s, \ \forall t \geq 0, \\ d_5) \quad \int_s^t \phi^o(X_r, v(r)) \geq \int_s^t \langle dK_r, v(r)\rangle, \ \forall v \in C\left([0,T], \mathbb{R}^d\right), \ \forall 0 \leq s \leq t \leq T, \\ d_6) \quad \int_s^t \langle y(r) - X_r, dJ_r\rangle + \int_s^t \varphi(X_r)\, dr \leq \int_s^t \varphi(y(r))\, dr, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall y \in C\left([0,T], \mathbb{R}^d\right), \ \forall 0 \leq s \leq t \leq T. \end{cases}$$

Theorem 41 *(Uniqueness) Let the assumptions (49), (51) and (57) be satisfied and suppose that* (X, J, K, β), $(\hat{X}, \hat{J}, \hat{K}, \hat{\beta})$ *are two solutions of the SVI (56), corresponding, respectively, to the initial conditions* $\xi, \hat{\xi} \in L^0(\Omega, \mathcal{F}_0, \mathbb{P}; \overline{D(\varphi)})$. *Letting* $p \geq 1$, *if* $\xi = \hat{\xi}$, $\mathbb{P}-a.s.\ \omega \in \Omega$, *then*

$$X_s = \hat{X}_s, \ J_s = \hat{J}_s, \ K_s = \hat{K}_s, \ \beta_s = \hat{\beta}_s, \quad \text{for all } s \in [0,T].$$

Proposition 42 *Let* $\xi \in L^0(\Omega, \mathcal{F}_0, P; \overline{D(\varphi)})$. *If* $M \in S_d^0[0,T]$, $M_0 = 0$, *then the stochastic differential equation*

$$\begin{cases} dX_t + \partial\varphi(X_t)\, dt + \partial_C \phi(X_t)\, dt + G(t, X_t)dt \ni dM_t, \quad t \in [0,T], \\ X_0 = \xi. \end{cases} \tag{58}$$

admits a unique solution $(X, J, K, \beta) \in S_d^0[0,T] \times S_d^0[0,T] \times S_d^0[0,T] \times L^0\left(\Omega; L^1\left([0,T]; \mathbb{R}^d\right)\right)$. *In the particular case when the martingale* M *is an Itô integral (i.e.* $M_t = \int_0^t Q_s dB_s$), *if there exist* $p \geq 2$ *and* $u_0 \in int\,(Dom\,(\partial(\varphi)))$ *satisfying*

$$\mathbb{E}|\xi|^p + \mathbb{E}\left(\int_0^T |\beta_{t,u_0}|\, dt\right)^p + \mathbb{E}\left(\int_0^T |Q(t, u_0)|^2\, dt\right)^{p/2} < +\infty, \tag{59}$$

with $\beta_{t,u_0} \in G(t, u_0)$, *then*

$$(X, K, \beta) \in S_d^p[0,T] \times (S_d^{p/2}[0,T] \cap L^{p/2}(\Omega; BV\left([0,T], \mathbb{R}^d\right))) \times L^p\left(\Omega; L^1\left([0,T]; \mathbb{R}^d\right)\right).$$

Theorem 43 *If assumptions (49,), (51) and (57) are satisfied, then the SVI (56) has a unique solution* $(X, J, K, \beta) \in S_d^0[0,T] \times (S_d^0[0,T])^2 \times L^0\left(\Omega; L^1\left([0,T]; \mathbb{R}^d\right)\right)$. *Moreover, if there exist* $p \geq 2$ *and* $u_0 \in int\,(Dom\,(\partial\varphi))$ *such that, for all* $T \geq 0$,

$$\mathbb{E}|\xi|^p + \mathbb{E}\left(\int_0^T |\beta_{t,u_0}|\, dt\right)^p + \mathbb{E}\left(\int_0^T |Q_t|^2\, dt\right)^{p/2} < +\infty, \tag{60}$$

then $(X, K) \in S_d^p[0,T] \times (S_d^{p/2}[0,T] \cap L^{p/2}\left(\Omega; BV\left([0,T], \mathbb{R}^d\right)\right))$.

6 Annex

6.1 Absolutely continuous and bounded variation functions

We present in this section some definitions and important results concerning absolutely continuous and bounded variation functions (for more details see Brézis [10]).

6.2 A priori estimates for the generalized Skorohod problem

We give four lemmas with a priori estimates of the solution $(x,k) \in \mathcal{SP}(H\partial\varphi; x_0, m)$. These lemmas and also their proofs are similar with those from the monograph of Pardoux & Răşcanu [41]. The three first lemmas are used to prove the main key Lemma 47.

Lemma 44 *Let the assumptions (6), (7), (5) and (9) be satisfied. If $(x,k) \in \mathcal{SP}(H\partial\varphi; x_0, m)$, then for all $0 \le s \le t \le T$:*

$$\begin{aligned}\mathbf{m}_x(t-s) \le & \left[(t-s) + \mathbf{m}_m(t-s) + \sqrt{\mathbf{m}_m(t-s)\left(\updownarrow k\updownarrow_t - \updownarrow k\updownarrow_s\right)}\right] \\ & \times \exp\left\{C\left[1 + (t-s) + \left(\updownarrow k\updownarrow_t - \updownarrow k\updownarrow_s + 1\right)\left(\updownarrow k\updownarrow_t - \updownarrow k\updownarrow_s\right)\right]\right\},\end{aligned} \tag{61}$$

where $C = C(b,c,L) > 0$.

Lemma 45 *Let the assumptions (6), (7), (5) and (9) be satisfied. If $(x,k) \in \mathcal{SP}(H\partial\varphi; x_0, m)$, $0 \le s \le t \le T$ and*

$$\sup_{r\in[s,t]} |x(r) - x(s)| \le 2\delta_0 = \frac{\rho_0}{2bc} \wedge \rho_0\,, \quad \text{with } \rho_0 = \frac{r_0}{2(1 + r_0 + h_0)},$$

then

$$\updownarrow k\updownarrow_t - \updownarrow k\updownarrow_s \le \frac{1}{\rho_0}|k(t) - k(s)| + \frac{3L}{\rho_0}(t-s) \tag{62}$$

and

$$|x(t) - x(s)| + \updownarrow k\updownarrow_t - \updownarrow k\updownarrow_s \le \sqrt{t - s + \mathbf{m}_m(t-s)} \times e^{C_T\left(1 + \|m\|_T^2\right)}, \tag{63}$$

where $C_T = C(b,c,r_0,h_0,L,T) > 0$.

Lemma 46 *Let the assumptions (6), (7), (5) and (9) be satisfied. Let $(x,k) \in \mathcal{SP}(H\partial\varphi; x_0, m)$, $0 \le s \le t \le T$ and $x(r) \in D_{\delta_0}$, for all $r \in [s,t]$. Then*

$$\updownarrow k\updownarrow_t - \updownarrow k\updownarrow_s \le L\left(1 + \frac{2}{\delta_0}\right)(t-s)$$

and

$$\mathbf{m}_x(t-s) \le C_T \times \left[(t-s) + \mathbf{m}_m(t-s)\right],$$

where $C_T = C_T(b,c,r_0,h_0,L,T) > 0$.

Lemma 47 *Let the assumptions (6), (7), (5) and (9) be satisfied and $(x,k) \in \mathcal{SP}(H\partial\varphi; x_0, m)$. There exists a positive constant $C_T(\|m\|_T) = C(x_0,b,c,r_0,h_0,L,T,\|m\|_T)$, increasing function with respect to $\|m\|_T$, such that, for all $0 \le s \le t \le T$:*

$$\begin{aligned} &(a) \qquad \|x\|_T + \updownarrow k\updownarrow_T \le C_T(\|m\|_T), \\ &(b) \qquad |x(t) - x(s)| + \updownarrow k\updownarrow_t - \updownarrow k\updownarrow_s \le C_T(\|m\|_T) \times \sqrt{\mu_m(t-s)}. \end{aligned} \tag{64}$$

6.3 Useful inequalities

This section provides some useful deterministic and stochastic inequalities which are used in our studies, mainly in Chapter 2, Chapter 3 and Chapter 4.

6.3.1 Deterministic and forward case inequalities

Proposition 48 *Let $x \in BV_{loc}\left([0,\infty);\mathbb{R}^d\right)$ and $V \in BV_{loc}([0,\infty);\mathbb{R})$ be continuous functions. Let R, $N : [0,\infty) \to [0,\infty)$ be two continuous increasing functions. If*

$$\langle x(t), dx(t)\rangle \le dR(t) + |x(t)|\, dN(t) + |x(t)|^2\, dV(t)$$

as signed measures on $[0,\infty)$, then for all $0 \le t \le T$,

$$\left\|e^{-V}x\right\|_{[t,T]} \le 2\left[\left|e^{-V(t)}x(t)\right| + \left(\int_t^T e^{-2V(s)}dR(s)\right)^{1/2} + \int_t^T e^{-V(s)}dN(s)\right]. \tag{65}$$

If $R = 0$ then, for all $0 \le t \le s$,

$$|x(s)| \le e^{V(s)-V(t)}|x(t)| + \int_t^s e^{V(s)-V(r)}dN(r). \tag{66}$$

Recall, from Pardoux & Răşcanu [41], an estimate on the local semimartingale $X \in S_d^0$ of the form

$$X_t = X_0 + K_t + \int_0^t G_s dB_s,\ \ t \ge 0, \quad \mathbb{P}-a.s.\ \omega \in \Omega, \tag{67}$$

where $K \in S_d^0,\ \ K \in BV_{loc}\left([0,\infty);\mathbb{R}^d\right),\ \ K_0 = 0,\ \mathbb{P}-a.s.\ \omega \in \Omega$ and $G \in \Lambda^0_{d\times k}$.

For $p \ge 1$ denote $m_p \overset{def}{=} 1 \vee (p-1)$ and we have the following result.

Proposition 49 *Let $X \in S_d^0$ be a local semimartingale of the form (67). Assume there exist $p \ge 1$ and V a $\mathcal{P}-$m.b-v.c.s.p., $V_0 = 0$, such that as signed measures on $[0,\infty[$:*

$$\langle X_t, dK_t\rangle + \frac{1}{2}m_p\,|G_t|^2\,dt \le |X_t|^2 dV_t,\ \ \mathbb{P}-a.s.\ \omega \in \Omega. \tag{68}$$

Then, for all $\delta \ge 0$, $0 \le t \le s$, we have that

$$\mathbb{E}^{\mathcal{F}_t}\frac{\left|e^{-V_s}X_s\right|^p}{\left(1+\delta\left|e^{-V_s}X_s\right|^2\right)^{p/2}} \le \frac{\left|e^{-V_t}X_t\right|^p}{\left(1+\delta\left|e^{-V_t}X_t\right|^2\right)^{p/2}},\ \mathbb{P}-a.s.\ \omega \in \Omega. \tag{69}$$

Proposition 50 *Let $X \in S_d^0$ be a local semimartingale of the form (67). Assume there exist three $\mathcal{P}-$measurable, increasing, continuous, stochastic processes D, R, N, $p \ge 1$ and a $\mathcal{P}-$measurable, with bounded variation, continuous stochastic process V, $D_0 = R_0 = N_0 = V_0 = 0$, such that, as signed measures on $[0,\infty[$, $\mathbb{P}-a.s.\ \omega \in \Omega$,*

$$dD_t + \langle X_t, dK_t\rangle + (\frac{1}{2}m_p + 9p\lambda)\,|G_t|^2\,dt \le \mathbf{1}_{p\ge 2}dR_t + |X_t|dJ_t + |X_t|^2 dV_t.$$

Then, for all $0 \le t \le s$*, we have,* $\mathbb{P}-a.s.\ \omega\in\Omega$*,*

$$\begin{aligned}\mathbb{E}^{\mathcal{F}_t}||e^{-V}X||^p_{[t,s]}+\mathbb{E}^{\mathcal{F}_t}\int_t^s e^{-pV_r}|X_r|^{p-2}dD_r+\mathbb{E}^{\mathcal{F}_t}\left(\int_t^s e^{-2V_r}(dD_r+|G_r|^2dr)\right)^{\frac{p}{2}}\\ \le C_{p,\lambda}\left[e^{-pV_t}|X_t|^p+\mathbb{E}^{\mathcal{F}_t}\left(\int_t^s e^{-2V_r}\mathbf{1}_{p\ge2}dR_t\right)^{\frac{p}{2}}+\mathbb{E}^{\mathcal{F}_t}\left(\int_t^s e^{-V_r}dN_r\right)^p\right].\end{aligned}\tag{70}$$

6.3.2 Backward case inequalities

We shall derive some important estimations on the stochastic processes $(Y,Z)\in S_d^0[0,T]\times\Lambda^0_{d\times k}(0,T)$ satisfying for all $t\in[0,T]$, $\mathbb{P}-a.s.\ \omega\in\Omega$,

$$Y_t=Y_T+\int_t^T dK_s-\int_t^T Z_sdB_s,$$

with $K\in S_d^0$ be such that $K_\cdot(\omega)\in BV_{loc}\left([0,\infty);\mathbb{R}^d\right),\ \mathbb{P}-a.s.\ \omega\in\Omega$. For more details concerning the results found in this section one can consult Section 6.3.4 from Pardoux and Răşcanu [41].

A fundamental inequality

Let $(Y,Z)\in S_d^0[0,T]\times\Lambda^0_{d\times k}(0,T)$ satisfying an identity of the form

$$Y_t=Y_T+\int_t^T dK_s-\int_t^T Z_sdB_s,\qquad t\in[0,T],\quad \mathbb{P}-a.s.\ \omega\in\Omega,\tag{71}$$

where $K\in S_d^0([0,T])$ and $K_\cdot(\omega)\in BV\left([0,T];\mathbb{R}^d\right),\ \mathbb{P}-a.s.\ \omega\in\Omega$.

Assume there exist

- D,R,N - three progressively measurable increasing continuous stochastic processes with $D_0=R_0=N_0=0$,
- V - a progressively measurable bounded variation continuous stochastic process with $V_0=0$,
- $0\le\lambda<1<p$,

such that, as measures on $[0,T]$, $\mathbb{P}-a.s.\ \omega\in\Omega$,

$$dD_t+\langle Y_t,dK_t\rangle\le\left[\mathbf{1}_{p\ge2}dR_t+|Y_t|dN_t+|Y_t|^2dV_t\right]+\frac{n_p}{2}\lambda\,|Z_t|^2\,dt,\tag{72}$$

where $n_p\overset{def}{=}1\wedge(p-1)$. Proposition 6.80 from Pardoux and Răşcanu [41] yields the following important result.

Proposition 51 *If (71) and (72) hold, and moreover*

$$\mathbb{E}\left\|Ye^V\right\|_T^p < \infty,$$

then there exists a positive constant $C_{p,\lambda}$, depending only upon (p,λ), such that, $\mathbb{P}-a.s.\ \omega\in\Omega$, for all $t\in[0,T]$,

$$\begin{aligned}&\mathbb{E}^{\mathcal{F}_t}\sup_{s\in[t,T]}\left|e^{V_s}Y_s\right|^p+\mathbb{E}^{\mathcal{F}_t}\left(\int_t^T e^{2V_s}dD_s\right)^{p/2}+\mathbb{E}^{\mathcal{F}_t}\left(\int_t^T e^{2V_s}\left|Z_s\right|^2ds\right)^{p/2}\\ &\quad+\mathbb{E}^{\mathcal{F}_t}\int_t^T e^{pV_s}\left|Y_s\right|^{p-2}\mathbf{1}_{Y_s\neq 0}dD_s+\mathbb{E}^{\mathcal{F}_t}\int_t^T e^{pV_s}\left|Y_s\right|^{p-2}\mathbf{1}_{Y_s\neq 0}\left|Z_s\right|^2ds\\ &\leq C_{p,\lambda}\,\mathbb{E}^{\mathcal{F}_t}\left[\left|e^{V_T}Y_T\right|^p+\left(\int_t^T e^{2V_s}\mathbf{1}_{p\geq 2}dR_s\right)^{p/2}+\left(\int_t^T e^{V_s}dN_s\right)^p\right].\end{aligned}\tag{73}$$

In addition, if $R=N=0$, then, for all $t\in[0,T]$,

$$e^{pV_t}\left|Y_t\right|^p\leq\mathbb{E}^{\mathcal{F}_t}e^{pV_T}\left|Y_T\right|^p,\quad \mathbb{P}-a.s.\ \omega\in\Omega.\tag{74}$$

Proposition 52 (See Proposition 6.69 from [41]) *Let $\delta\in\{-1,1\}$ and consider $Y,K,A:\Omega\times\mathbb{R}_+\to\mathbb{R}$ and $G:\Omega\times\mathbb{R}_+\to\mathbb{R}^k$ four progressively measurable stochastic processes such that*

$$\begin{aligned}&i)\quad && Y,K,A \text{ are continuous stochastic processes},\\ &ii)\quad && A_\cdot,K_\cdot\in BV_{loc}\left([0,\infty[\,;\mathbb{R}\right),\ A_0=K_0=0,\ \mathbb{P}-a.s.\ \omega\in\Omega,\\ &iii)\quad && \int_t^s\left|G_r\right|^2dr<\infty,\ \mathbb{P}-a.s.\ \omega\in\Omega,\ \forall 0\leq t\leq s.\end{aligned}$$

If, for all $0\leq t\leq s$,

$$\delta\left(Y_t-Y_s\right)\leq\int_t^s\left(dK_r+Y_rdA_r\right)+\int_t^s\left\langle G_r,dB_r\right\rangle,\quad \mathbb{P}-a.s.\ \omega\in\Omega,$$

then

$$\delta\left(Y_te^{\delta A_t}-Y_se^{\delta A_s}\right)\leq\int_t^s e^{\delta A_r}dK_r+\int_t^s e^{\delta A_r}\left\langle G_r,dB_r\right\rangle,\quad \mathbb{P}-a.s.\ \omega\in\Omega.$$

6.4 Tightness results

The next results (found in the monograph [41]) are mainly used in Chapter 2 and Chapter 3 and are useful tools when we prove the existence of some weak solutions for our equations.

Proposition 53 *Let $\{X_t^n:t\geq 0\}$, $n\in\mathbb{N}^*$, be a family of $\mathbb{R}^d-$valued continuous stochastic processes defined on probability space $(\Omega,\mathcal{F},\mathbb{P})$. Suppose that, for every $T\geq 0$, there exist $\alpha=\alpha_T>0$ and $b=b_T\in C\left(\mathbb{R}_+\right)$ with $b(0)=0$ (both independent of n), such that*

$$\begin{aligned}&(i)\quad && \lim_{N\to\infty}\left[\sup_{n\in\mathbb{N}^*}\mathbb{P}(\{|X_0^n|\geq N\})\right]=0,\\ &(ii)\quad && \mathbb{E}\left[1\wedge\sup_{0\leq s\leq\varepsilon}\left|X_{t+s}^n-X_t^n\right|^\alpha\right]\leq\varepsilon\cdot b(\varepsilon),\ \forall\,\varepsilon>0,n\geq 1,\ t\in[0,T].\end{aligned}$$

Then $\{X^n:n\in\mathbb{N}^*\}$ is tight in $C(\mathbb{R}_+;\mathbb{R}^d)$.

Proposition 54 *Let $X,\hat{X}\in S_d^0\left[0,T\right]$ and $B,\hat{B}$ be two $\mathbb{R}^k-$Brownian motions and $g:\mathbb{R}_+\times\mathbb{R}^d\rightarrow\mathbb{R}^{d\times k}$ be a function satisfying*

$$g\left(\cdot,y\right)\text{ is measurable }\forall\, y\in\mathbb{R}^d,\quad\text{and}$$
$$y\mapsto g\left(t,y\right)\text{ is continuous } dt-a.e..$$

If

$$\mathcal{L}\left(X,B\right)=\mathcal{L}(\hat{X},\hat{B}),\quad\text{on } C(\mathbb{R}_+,\mathbb{R}^{d+k}),$$

then

$$\mathcal{L}\left(X,B,\int_0^{\cdot}g\left(s,X_s\right)dB_s\right)=\mathcal{L}\left(\hat{X},\hat{B},\int_0^{\cdot}g\left(s,\hat{X}_s\right)d\hat{B}_s\right),\quad\text{on } C(\mathbb{R}_+,\mathbb{R}^{d+k+d}).$$

Lemma 55 *Let $g:\mathbb{R}_+\rightarrow\mathbb{R}_+$ be a continuous function satisfying $g\left(0\right)=0$ and $G:C\left(\mathbb{R}_+;\mathbb{R}^d\right)\rightarrow\mathbb{R}_+$ be a mapping which is bounded on compact subsets of $C\left(\mathbb{R}_+;\mathbb{R}^d\right)$. Let X^n,Y^n, $n\in\mathbb{N}^*$, be random variables with values in $C\left(\mathbb{R}_+;\mathbb{R}^d\right)$. If $\{Y^n:n\in\mathbb{N}^*\}$ is tight and, for all $n\in\mathbb{N}^*$,*

$$\begin{aligned}(i)&\qquad |X_0^n|\leq G\left(Y^n\right),\ a.s.,\\(ii)&\qquad \mathbf{m}_{X^n}\left(\varepsilon;[0,T]\right)\leq G\left(Y^n\right)g\left(\mathbf{m}_{Y^n}\left(\varepsilon;[0,T]\right)\right),\ a.s.,\quad\forall\,\varepsilon,T>0,\end{aligned}$$

then $\{X^n:n\in\mathbb{N}^*\}$ is tight.

Lemma 56 *Let B, B^n, $\bar{B}^n:\Omega\times[0,\infty)\rightarrow\mathbb{R}^k$ and X, X^n, $\bar{X}^n:\Omega\times[0,\infty)\rightarrow\mathbb{R}^{d\times k}$ be continuous stochastic processes such that*

$$(i)\qquad B^n\text{ is }\mathcal{F}_t^{B^n,X^n}-\text{Brownian motion, for all } n\geq 1,$$

$$(ii)\qquad \mathcal{L}(X^n,B^n)=\mathcal{L}\big(\bar{X}^n,\bar{B}^n\big)\text{ on } C(\mathbb{R}_+,\mathbb{R}^{d\times k}\times\mathbb{R}^k),\text{ for all } n\geq 1,$$

$$(iii)\qquad \int_0^T\left|\bar{X}_s^n-\bar{X}_s\right|^2ds+\sup_{t\in[0,T]}\left|\bar{B}_t^n-\bar{B}_t\right|\longrightarrow 0\text{ in probability, as } n\rightarrow\infty,\text{ for all } T>0.$$

Then $(\bar{B}^n,\{\mathcal{F}_t^{\bar{B}^n,\bar{X}^n}\}),n\geq 1$, and $(\bar{B},\{\mathcal{F}_t^{\bar{B},\bar{X}}\})$ are Brownian motions and, as $n\rightarrow\infty$,

$$\sup_{t\in[0,T]}\left|\int_0^t\bar{X}_s^n d\bar{B}_s^n-\int_0^t\bar{X}_s d\bar{B}_s\right|\underset{n\rightarrow\infty}{\longrightarrow}0,\quad\text{in probability.}$$

References

[1] Asiminoaei, I.; Răşcanu, A., *Approximation and simulation of stochastic variational inequalities-splitting up method*, Numer. Funct. Anal. and Optim. 18, no. 3&4, pp. 251-282, 1997.

[2] Attouch, H. ; Damlamian, A., *On multivalued evolution equations in Hilbert spaces*, Israel J. Math 12, pp 373-390, 1972.

[3] Barbu, V., *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff, Leyden, 1975.

[4] Barbu, V.; Răşcanu, A., *Parabolic Variational Inequalities with singular inputs*, Differential and Integral Equation, Vol. 10, no. 1, pp. 67-83, 1997.

[5] Bensoussan, A.; Răşcanu, A., *Stochastic variational inequalities in infinite-dimensional spaces*, Numer. Funct. Anal. Optim. 18, no. 1-2, pp. 19-54, 1997.

[6] Bernard, A.; El Kharroubi, A., *Régulations déterministes et stochastiques dans le premier "orthant" de* $\mathbb{R}^n$, Stochastics Stochastics Rep. 34, pp. 149-167, 1991.

[7] Billingsley, P. , *Convergence of Probability Measures*, New York, NY: John Wiley & Sons, 1999.

[8] Bismut, J.M., *Conjugate Convex Functions in Optimal Stochastic Control*, J. Math. Anal. Appl.,44, pp. 384-404, 1973.

[9] Boufoussi, B.; van Casteren, J., *An approximation result for a nonlinear Neumann boundary value problem via BSDEs*, Stochastic Processes and their Applications, Volume 114, pp. 331-350, 2004.

[10] Brézis, H. - *Opérateurs Maximaux Monotones et Semigroupes de Contractions Dans les Espaces de Hilbert*, North-Holland, Amsterdam, 1973.

[11] Buckdahn, R.; Engelbert, H.-J.; Răşcanu, A., *On weak solutions of backward stochastic differential equations*, Teor. Veroyatn. Primen. 49, 1, pp. 70-108, 2004.

[12] Buckdahn, R.; Răşcanu, A., *On the existence of stochastic optimal control of distributed state system*, Nonlinear Anal. 52, no. 4, pp. 1153-1184, 2003.

[13] Buckdahn, R.; Quincampoix, M.; Răşcanu, A., *Viability property for a backward stochastic differential equation and applications to partial differential equations*, Probab. Theory Related Fields 116, no. 4, pp. 485-504, 2000.

[14] Cépa, E., *Equations différentielles stochastiques multivoques. (French) [Multivalued stochastic differential equations]*, Séminaire de Probabilités, XXIX, Lecture Notes in Math., Springer, Berlin, pp. 86-107, 1995.

[15] Cépa, E., *Problème de Skorohod multivoque. (French) [Multivalued Skorohod problem]*, Ann. Probab. 26, no 2, pp. 500-532, 1998.

[16] Colombo, G; Goncharov, V., *Variational Inequalities and Regularity Properties of Closed Sets in Hilbert Spaces*, Journal of Convex Analysis, 8 (1), pp. 197-221, 2001.

[17] Constantini, C., *The Skorohod oblique reflection problem in domains with corners and application to stochastic differential equations*, Probab. Theory Relat. Fields 91, pp. 43-70, 1992.

[18] Degiovanni, M; Marino, A; Tosques, M, *Evolution Equations with Lack of Convexity*, Nonlinear Anal., 9 (2), pp. 1401-1443, 1985.

[19] Dupuis, P.; Ishii, H., *On Lipschitz continuity of the solution mapping to the Skorokhod problem, with applications*, Stochastics Stochastics Rep. 35, pp. 31-62, 1991.

[20] Dupuis, P.; Ishii, H., *SDEs with oblique reflection on nonsmooth domains*, Ann. Probab. 21, no. 1, pp. 554-580, 1993.

[21] Friedman, A., *Stochastic Differential Equations and Applications I*, Academic Press, 1975.

[22] Gassous, M., A., *Skorohod problem driven by Clarke subdifferential operator: Application to SDE*, accepted for publication at An Stiint U Al I-Mat, 2013.

[23] Gassous, M., A.; Răşcanu. A.; Rotenstein, E., *Stochastic variational inequalities with oblique subgradients*, Stochastic Processes and their Applications, Volume 122, Issue 7, pp. 2668-2700, 2012.

[24] Gassous, M., A.; Răşcanu. A.; Rotenstein, E., *Backward stochastic variational inequalities with oblique subgradients*, 30 pp., submitted for publication at Probability Theory and Related Fields, 2013.

[25] Gassous, M., A.; Răşcanu. A.; Rotenstein, E., *Multivalued differential equations with oblique subgradients on non-convex domains*, 20 pp., preprint, 2013.

[26] Hu, Y.; Tang, S. - *Multi-dimensional BSDE with oblique reflection and optimal switching*, Probability Theory and Related Fields, Volume 147, Issue 1-2, pp. 89-121, 2010.

[27] Jakubowski, A., *A non-Skorokhod topology on the Skorokhod space*, Electron. J. Probab. 2 (4), pp. 1-21, 1997.

[28] Karatzas, I; Shreve, S.E., *Brownian Motion and Stochastic Calculus*, Springer-Verlag New York Berlin, 2nd edition, 2005.

[29] LeJay, A., *BSDE driven by Dirichlet process and semi-linear parabolic PDE. Application to homogenization*, Stochastic Processes and Their Applications (1), pp. 1-39, 2002.

[30] Lions, P.L.; Sznitman, A., *Stochastic differential equations with reflecting boundary conditions*, Comm. Pure Appl. Math. 37, no. 4, pp. 511-537, 1984.

[31] Maticiuc, L.; Răşcanu, A., *Backward stochastic generalized variational inequality*, Applied analysis and differential equations, pp. 217-226, World Sci. Publ., Hackensack, NJ, 2007.

[32] Marino, A; Tosques, M., *Some Variational Problems with Lack of Convexity* in Methods of Nonconvex Analysis, Proceedings from CIME, Varenna 1989 (A. Cellina, ed.), Springer, Berlin, 1990.

[33] McKean, H., *A. Skorohod's stochastic integral equation for a reflecting barrier diffusion*, J. Math. Kyoto Univ., Volume 3, Number 1, pp. 85-88, 1963.

[34] Nualart, D.; Răşcanu, A., *Differential equations driven by fractional Brownian motion,* Collect. Math. 53, no. 1, pp. 55-81, 2002.

[35] Önskog, T.; Nyström, K., *Weak approximation of obliquely reflected diffusions in time-dependent domains*, Journal of Computational Mathematics, Volume 28, Issue 5, pp. 579-605, 2010.

[36] Pardoux, E., *BSDEs, weak convergence and homogenization of semilinear PDEs*, Nonlinear Analysis, Differential Equations and Control, NATO Science Series Volume 528, pp 503-549, 1999.

[37] Pardoux, E.; Peng, S., *Solution of a Backward Stochastic Differential Equation*, Systems and Control Letters, 14, pp. 55-61, 1990.

[38] Pardoux, E.; Peng, S., *Backward SDE's and quasilinear parabolic PDE's*, Stochastic PDE and Their Applications, LNCIS 176, Springer, pp. 200-217, 1992.

[39] Pardoux, E.; Răşcanu, A., *Backward stochastic differential equations with subdifferential operator and related variational inequalities*, Stochastic Processes and Their Applications, vol. 76, no. 2, pp. 191-215, 1998.

[40] Pardoux, E.; Răşcanu, A., *Backward stochastic variational inequalities*, Stochastics Stochastics Rep., 67(3-4), pp. 159-167, 1999.

[41] Pardoux, E.; Răşcanu, A., *SDEs, BSDEs and PDEs,* book, accepted for publication in Springer, 2013.

[42] Qin, S. ; Xue, X. , *Evolution inclusions with Clarke subdifferential type in Hilbert space,* Mathematical and Computer Modelling 51, pp 550-561, 2010.

[43] Răşcanu, A., *Deterministic and stochastic differential equations in Hilbert spaces involving multivalued maximal monotone operators*, Panamer. Math. J. 6, no. 3, pp. 83-119, 1996.

[44] Răşcanu, A.; Rotenstein, E., *The Fitzpatrick function-a bridge between convex analysis and multivalued stochastic differential equations*, J. Convex Anal. 18 (1), pp. 105-138, 2011.

[45] On the maximality of the sums of nonlinear monotone operators, Trans. Amer. Math. Soc. 149, pp. 75-88, 1970

[46] Rossi, R.; Savaré, G., *Existence and approximation results for gradient flows*, Atti Accad. Naz. Lincei Rendl. Cl. Sci. Fis. Mat. Natur., IX. Ser., Mat. Appl., 15 (3-4), pp. 183-196, 2004.

[47] Rossi, R.; Savaré, G., *Gradient Flows of Non Convex Functionals in Hilbert Spaces and Applications*, ESAIM Control Optim. Calc. Var., 12 (3), pp. 564-614, 2006.

[48] Rozkosz, A.; Słomiński, L., *On stability and existence of solutions of SDEs with reflection at the boundary*, Stochastic Processes Appl. 68, No.2, pp. 285-302, 1997.

[49] Saisho, Y., *Stochastic differential equations for multidimensional domain with reflecting boundary*, Probab. Theory Related Fields 74, pp. 455-477, 1987.

[50] Schirotzek, W. , *Nonsmooth Analysis*, Springer-Verlag, Berlin Heidelberg, 2007

[51] Słomiński, L., *On existence, uniqueness and stability of solutions of multidimensional SDEs with reflecting boundary conditions*, Ann. Inst. Henri Poincaré, Probab. Stat. 29 (2), pp. 163-198, 1993.

[52] Skorohod, A., *Stochastic equations for diffusion processes in a bounded region*, Veroyatnost. i Primenen. no. 6, pp. 264-274, 1961; no. 7, pp. 3-23, 1962.

[53] Tanaka, H., *Stochastic differential equations with reflecting boundary condition in convex regions*, Hiroshima Math. J., pp. 163-177, 1979.

[54] Zălinescu, C., *Programare matematică în spaţii normate infinit dimensionale*, Editura Academiei Române, Bucureşti, 1995.

[55] Zălinescu, C., *Convex analysis in general vector spaces*, World Scientific Publishing Co., New York, 2002.

[56] Zhang, J., *A numerical scheme for BSDEs*, Annals of Applied Probability, 14, pp. 459-488, 2004.