Stabilization of the Navier-Stokes equations

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Preface

The mathematical model which describes the evolution of a fluid is one of the most studied by the researchers because of its major importance. The governing equations are the well-known Navier-Stokes equations.

In the first chapter, we study the evolution of a fluid in a semi-infinite rectangle. Main results obtained concerning the stabilization of the Poiseuille parabolic profile were obtained by Krstic and his coworkers in [3, 85, 54, 85, 27, 77, 64]. We obtain a stabilizing feedback controller which acts only on the normal component of the velocity field, on the upper wall. The stability is guaranteed without any a priori condition on the viscosity coefficient. The controller is easily manageable from computational point of view since the operators in its form satisfy Riccati algebraic equations associated to parabolic equations on (0,1) with the same structure for different values of the Fourier modes.

The second chapter propose an nonlinear internal control for the Navier-Stokes equations with slip-boundary conditions. Moreover, we show that this controller steers the initial data into the space consisting of stable modes, in finite time.

Finally, the last chapter study the problem in finding a feedback controller which stabilizes, not only one steady-state solution, but a finite-set of steady-state solutions for the Navier-Stokes equations, by using the already known results obtained in [?]

The paper ends with an Appendix which contains a short introduction in the theoretical results used during the presentation.
Chapter 1

Stabilization of Navier-Stokes equations in a channel

In this chapter, we study the mathematical model which describes a Newtonian incompressible fluid evolving in a semi-infinite 2-D channel

\[(x, y) \in (-\infty, +\infty) \times (0, 1),\]

and 3-D channel

\[(x, y, z) \in (-\infty, +\infty) \times (0, 1) \times (-\infty, +\infty),\]

respectively. We shall consider both cases, when there is no action of magnetic type and the case when an external constant magnetic field acts transversal on the channel. The equations which govern the dynamics are the Navier-Stokes equations, and the MHD equations, a combination between the Navier-Stokes equations and Maxwell equations.

The both experimental and numerical analysis show that, for high values of Reynolds number (that is, for low values of the viscosity coefficient) the fluid may develop chaotic and turbulent movements. One of the principal mathematical tools used in order to attenuate or even eliminate the turbulence is the stabilization of the Navier-Stokes equations. In this chapter, we shall design a stabilizing feedback controller, which acts on the upper wall, on the normal component of the velocity field or on the tangential one. The stability is achieved without any a priori assumptions on the Reynolds number. We shall use the method of Fourier decomposition of the linearized system and reduction of the pressure, developed by Barbu in [14], to obtain an infinite parabolic system. The stabilization is achieved at each level, by using the decomposition method of the system in the stable and unstable part, developed by Barbu and Triggiani in [10]. The feedback form of the stabilizing controller is obtained via minimization of a quadratic functional cost associated to a linear parabolic problem.

1.1 Normal feedback stabilization of periodic flows in a two-dimensional channel

This section entirely contains the original results obtained by the author in [66].
1.1.1 Presenting the problem

The 2-D Navier-Stokes equations in a periodic channel are given by

\[
\begin{align*}
  u_t - \nu \Delta u + uu_x + vu_y &= p_x, \\
  v_t - \nu \Delta v + uv_x + vv_y &= p_y, \\
  u_x + v_y &= 0, \\
  u(t, x, 0) &= u(t, x, 1) = 0, \quad v(t, x, 0) = 0, \quad v(t, x, 1) = \Psi(t, x), \\
  u(t, x + 2\pi, y) &= u(t, x, y), \quad v(t, x + 2\pi, y) = v(t, x, y), \\
  p(t, x + 2\pi, y) &= p(t, x, y), \\
  \forall t \geq 0, \ x \in \mathbb{R}, \ y \in (0, 1),
\end{align*}
\]

and initial data

\[
\begin{align*}
  u(0, x, y) &= u_0(x, y), \quad v(0, x, y) = v_0(x, y), \quad \forall x \in \mathbb{R}, \ y \in (0, 1).
\end{align*}
\]

Here \( u = u(t, x, y) \) and \( v = v(t, x, y) \) are the tangential component, normal component respectively, of the fluid; \( p = p(t, x, y) \) is the pressure and \( \nu \) is the viscosity coefficient.

The equilibrium solution which will be stabilized is the parabolic Poiseuille profile, given by

\[
U^e(y) = C(y^2 - y), \quad V^e \equiv 0,
\]

where \( C = -\frac{a}{2\nu}, \ a \in \mathbb{R}_+ \).

The linearization of (1.1.1) around the steady-state is given by

\[
\begin{align*}
  u_t - \nu \Delta u + u_x U^e + vU_y^e &= p_x, \\
  v_t - \nu \Delta v + v_x U^e &= p_y, \\
  u_x + v_y &= 0, \\
  u(t, x, 0) &= u(t, x, 1) = 0, \quad v(t, x, 0) = 0, \quad v(t, x, 1) = \Psi(t, x), \\
  u(t, x + 2\pi, y) &= u(t, x, y), \quad v(t, x + 2\pi, y) = v(t, x, y), \\
  p(t, x + 2\pi, y) &= p(t, x, y), \ \forall t \geq 0, \ x \in \mathbb{R}, \ y \in (0, 1),
\end{align*}
\]

and initial data

\[
\begin{align*}
  u(0, x, y) &= u^0(x, y) := u_0(x, y) - U^e(y), \quad v(0, x, y) = v^0(x, y) := v_0(x, y), \\
  \forall x \in \mathbb{R}, \ y \in (0, 1).
\end{align*}
\]

In what follows we shall design a normal finite-dimensional feedback controller \( \Psi \) which stabilizes the linearized equation (1.1.2)
1.1.2 Preliminaries

The Fourier functional setting, in which we shall work, is given by the space \( L_{2\pi}^2(Q), Q = (0, 2\pi) \times (0, 1) \), containing all the functions \( u \in L_{\text{loc}}^2(\mathbb{R} \times (0, 1)) \), which are \( 2\pi \)-periodic with respect to \( x \). These functions are characterized by their Fourier series

\[
u(x, y) = \sum_{k \in \mathbb{Z}} u_k(y)e^{ikx}, \quad u_k = \overline{u_{-k}}, \forall k \in \mathbb{Z},
\]

such that

\[
\sum_{k \in \mathbb{Z}} \int_0^1 |u_k(y)|^2 dy < \infty.
\]

The norm in \( L_{2\pi}^2(Q) \) is

\[
\|u\|_{L_{2\pi}^2(Q)} := \left( \sum_{k \in \mathbb{Z}} 2\pi \|u_k\|^2_{L^2(0, 1)} \right)^{\frac{1}{2}}.
\]

We shall denote by \( \| \cdot \| \) in both spaces \( L_{2\pi}^2(Q) \) and \( L^2(0, 1) \).

We consider \( H \) to be the complexified space of \( L^2(0, 1) \). We denote by \( \| \cdot \| \) the norm in \( H \), and by \( \langle \cdot, \cdot \rangle \) the scalar product.

Returning to system (1.1.2) we rewrite it in terms of Fourier modes of the velocity field, the pressure and the control, that is

\[
u = \sum_{k \in \mathbb{Z}} v_k(t, y)e^{ikx}, \quad \Psi = \sum_{k \in \mathbb{Z}} \psi_k(t)e^{ikx}.
\]

We obtain

\[
\begin{cases}
(u_k)_t - \nu u_k'' + (\nu k^2 + ik U^e)u_k + (U^e)'v_k = ikp_k \text{ a.p.t. in } (0, 1), \\
(v_k)_t - \nu v_k'' + (\nu k^2 + ik U^e)v_k = p_k' \text{ a.p.t. in } (0, 1), \\
\quad ikv_k' + v_k' = 0 \text{ a.p.t. in } (0, 1), \\
\quad u_k(0) = u_k(1) = 0, v_k(0) = 0, v_k(1) = \psi_k,
\end{cases}
\]

and initial data \( u_k^0, v_k^0 \), for all \( k \in \mathbb{Z} \). (We denote by \( ' \) the partial derivative with respect to \( y \), i.e., \( \frac{\partial}{\partial y} \).)

Next, the idea is to eliminate the pressure from the equations in the next manner: we derive (1.1.3) with respect to \( y \) and add the result to (1.1.3) multiplied by \( ik \). Using also the divergence free condition, we arrive at the next system, verified by \( v_k \)

\[
\begin{cases}
(-\nu v_k'' + k^2 v_k)_t + \nu v_k'' - (2\nu k^2 + ik U^e)v_k'' \\
\quad + k(\nu k^3 + ik^2 U^e + i(U^e)')v_k = 0, \forall t \geq 0, \forall y \in (0, 1), \\
v_k'(t, 0) = v_k'(t, 1) = 0, v_k(t, 0) = 0, v_k(t, 1) = \psi_k(t), \forall t \geq 0, \\
v_k(0, y) = v_k^0(y), y \in (0, 1).
\end{cases}
\]
We introduce the operators
\[ L_k : \mathcal{D}(L_k) \subset H \to H \quad \text{and} \quad F_k : \mathcal{D}(F_k) \subset H \to H, \]
defined as
\[
L_k v := -v'' + k^2 v, \quad \mathcal{D}(L_k) = H^2(0, 1) \cap H^1_0(0, 1), \tag{1.1.5}
\]
\[
F_k v := \nu v''' - (2\nu k^2 + i ku^c)v'' + k(\nu k^3 + ik^2 u^c + i(\nu' v') v, \tag{1.1.6}
\]
\[
\mathcal{D}(F_k) = H^4(0, 1) \cap H^2_0(0, 1).
\]
AS well, we introduce the next differential forms
\[ L_k v := -v'' + k^2 v, \]
\[ F_k v := \nu v''' - (2\nu k^2 + i ku^c)v'' + k(\nu k^3 + ik^2 u^c + i(\nu' v') v. \]
Then, system (1.1.4) can be written as
\[
(L_k(v_k - w_k))_t + (F_k L_k^{-1}) L_k(v_k - w_k) = \theta_k w_k - (L_k(w_k))_t, \tag{1.1.7}
\]
where \( w_k = w_k(t, y) \) satisfies
\[
\begin{cases}
\theta_k w_k + F_k w_k = 0, & t \geq 0, \ y \in (0, 1), \\
w'_k(0) = w'_k(1) = 0, w_k = 0, w_k(1) = \psi_k,
\end{cases} \tag{1.1.8}
\]
for some \( \theta_k > 0, \) sufficiently large. This suggests to introduce the next operators
\[ A_k : \mathcal{D}(A_k) \subset H \to H, \] for all \( k \in \mathbb{Z}^* , \]
defined as
\[ A_k := F_k L_k^{-1}, \quad \mathcal{D}(A_k) = \{ v \in H : L_k^{-1} v \in \mathcal{D}(F_k) \}, \tag{1.1.9}
\]
for which we have the next result, due to de Barbu [17].

**Lemma 1.1.1** For all \( k \in \mathbb{Z}^* , \) operator \(-A_k\) generates a \( C_0 \) analytic semigroup in \( H, \) and for all \( \lambda \in \rho(-A_k), \) \( (\lambda I + A_k)^{-1} \) is compact. Besides, we have
\[
\sigma(-A_k) \subset \{ \lambda \in \mathbb{C} : \Re \lambda \leq 0 \}, \forall |k| > S,
\]
where
\[
S = \frac{1}{\sqrt{\nu}} \left( 1 + \frac{a}{\sqrt{2\nu}} \right)^{\frac{1}{2}}. \tag{1.1.10}
\]
A first consequence of this lemma is: for all \( |k| > S, \) the solution to the system (1.1.4), with null control on the boundary, satisfies the next exponential decay
\[
\|u_k(t)\|^2 + \|v_k(t)\|^2 \leq e^{-\nu S^2 t} \left( \|u_k^0\|^2 + \|v_k^0\|^2 \right), \forall t \geq 0, \forall |k| > S. \tag{1.1.11}
\]
Therefore, it remains to control system (1.1.4) for \( 0 < |k| \leq S \) only.
After some computations it turns out that system (1.1.7) can be written equivalently in the next form

\[
\frac{d}{dt}(\tilde{L}_k v_k(t)) + \tilde{A}_k(\tilde{L}_k v_k(t)) = (\theta_k + \tilde{F}_k)(D_k \psi_k(t)), \quad t > 0.
\] (1.1.12)

Here \(\tilde{L}_k, \tilde{F}_k\) and \(\tilde{A}_k\) are the extensions to \(H\) of the operators \(L_k, F_k, A_k\), respectively; and \(D_k\) is the Dirichlet operator associated to \(\theta_k + \tilde{F}_k\). Equation (1.1.12) is understood in the next weak sense

\[
\left\langle \frac{d}{dt} L_k v_k(t), \phi \right\rangle + \left\langle L_k v_k, A_k^* \phi \right\rangle = \left\langle \psi_k(t), ((\theta_k + \tilde{F}_k)D_k)^* \phi \right\rangle, \quad \forall \phi \in \mathcal{D}(A_k^*),
\]

where the dual

\[
((\theta_k + \tilde{F}_k)D_k)^* \xi = \nu \xi'''(1),
\] (1.1.13)

for all \(\xi \in H^4(0,1), \xi(0) = \xi(1) = 0, \xi'(0) = \xi'(1) = 0\).

### 1.1.3 Further properties of the operators \(A_k\) and \(D_k\)

Appealing to Fredholm theory for compact operators, by Lemma 1.1.1, \(-A_k\) has a countable set of eigenvalues \(\{\lambda_j^k\}_{j=1}^\infty\); moreover, there is a finite number \(N_k\) of eigenvalues \(\lambda_j^k\) for which \(\Re \lambda_j^k \geq 0\), the unstable eigenvalues. We denote by \(\{\phi_j^k\}_{j=1}^\infty\) and \(\{\phi_j^{k*}\}_{j=1}^\infty\) the eigenfunctions of \(-A_k\) and \(-A_k^*\), respectively.

We have a result of ”unique continuation” type for the eigenfunctions of the dual operator \(-A_k^*\).

**Lema 1.1.2** Let \(\overline{\lambda_j^k}\), an unstable eigenvalue for the dual \(-A_k^*\). Then, we can assume that the corresponding eigenfunction \(\phi_j^{k*}\) can be chosen such that \(\Re(\phi_j^{k*})'''(1) > 0\).

Also, we have the next continuity result

**Proposition 1.1.1** For all \(0 < |k| \leq S\), \(D_k\) is continuous form \(\mathbb{C}\) to \(H\).

### 1.1.4 Feedback stabilization for the equivalent system (1.1.12)

For simplicity, we shall omit the symbol \(\tilde{\phantom{a}}\) and set

\[
z_k := L_k v_k, \quad B_k := (\theta_k + \tilde{F}_k)D_k.
\] (1.1.14)

Equation (1.1.12) becomes

\[
\begin{cases}
\frac{d}{dt}z_k(t) + A_k z_k(t) = B_k \psi_k(t), \quad t > 0, \\
z_k(0) = z_{0k},
\end{cases}
\] (1.1.15)

where \(z_{0k} = L_k v_{0k}^0\).

We denote by \(X_{N_k}^k := \text{linspan} \{\phi_j^k\}_{j=1}^{N_k}\) and by \(X_{N_k}^{k*} := \text{linspan} \{\phi_j^{k*}\}_{j=N_k+1}^\infty\). Then

\[
H = X_{N_k}^k \oplus X_{N_k}^{k*},
\]
as algebric summ. Introduce the projection $P_{N_k} : H \rightarrow X^u_{N_k}$, and its adjoint $P^*_{N_k}$, defined as

$$P_{N_k} := -\frac{1}{2\pi i} \int_{\Gamma} (\lambda I + A_k)^{-1} d\lambda$$

and

$$P^*_{N_k} := -\frac{1}{2\pi i} \int_{\Gamma} (\lambda I + A^*_k)^{-1} d\lambda.$$ 

Aslo, denote by

$$-A^u_{N_k} := P_{N_k}(-A_k)$$

and

$$-A^s_{N_k} := (I - P_{N_k})(-A_k),$$

where $-A^s_{N_k}$ satisfies the next exponential decay on $X^s_{N_k}$

$$\|e^{-tA^s_{N_k}}\|_{L(H,H)} \leq C_{\alpha_0} e^{-\alpha_0 t}, \forall t \geq 0,$$ (1.1.17)

for some $0 < \alpha_0 < |\Re \lambda_{N_k+1}|$.

Next, we decompose system (1.1.15) as

$$z_k = z_{N_k} + \zeta_{N_k},$$

where $z_{N_k} := P_{N_k} z_k$ and $\zeta_{N_k} := (I - P_{N_k}) z_k$,

by applying the operators $P_{N_k}$ and $I - P_{N_k}$ to the system (1.1.15), we get

\[\begin{align*}
\frac{d}{dt} z_{N_k} + A^u_{N_k} z_{N_k} &= P_{N_k}(B_k \psi_k), \\
z_{N_k}(0) &= P_{N_k} z_{0k},
\end{align*}\] (1.1.18)

on $X^u_{N_k}$;

\[\begin{align*}
\frac{d}{dt} \zeta_{N_k} + A^s_{N_k} \zeta_{N_k} &= (I - P_{N_k})(B_k \psi_k), \\
\zeta_{N_k}(0) &= (I - P_{N_k}) z_{0k},
\end{align*}\] (1.1.19)

respectively.

Because of the relation (1.1.17), system (1.1.19) is stable. Therefore, it remains to stabilize the finite-dimensional unstable system (1.1.18). We have

**Lemma 1.1.3** For all $0 < |k| \leq S$, there exist $\alpha_1, C_{\alpha_1} > 0$ and a control $\psi_k$ such that, once inserted into the system (1.1.18), the corresponding solution $z_{N_k}$ satisfies

$$\|z_{N_k}(t)\| \leq C_{\alpha_1} e^{-\alpha_1 t} \|z_{0k}\|, \forall t \geq 0.$$ 

Besides, the control can be choosen of class $C^1$, such that

$$\left|\frac{d}{dt} \psi_k(t)\right| + |\psi_k(t)| \leq C_{\alpha_1} e^{-\alpha_1 t} \|z_{0k}\|, \forall t \geq 0.$$ 

Using this result, we get

**Theorem 1.1.1** For all $0 < |k| \leq S$, there exists a control $\psi_k$ such that, once inserted into the system (1.1.15), the corresponding solution $z_k$ and the control satisfies the exponential decay

$$\left|\frac{d}{dt} \psi_k(t)\right| + |\psi_k(t)| \leq C_{\alpha_1} e^{-\alpha_1 t} \|z_{0k}\|, \|z_k(t)\| \leq C_{\alpha_0} e^{-\alpha_0 t} \|z_{0k}\|, \forall t \geq 0.$$
We derive a feedback form of the controller $\psi_k$ in Theorem 1.1.1 by using a classical approach: minimization of a cost function associated to the linear system (1.1.15), considered into the dual space $X = (H^2(0, 1) \cap H^1_0(0, 1))^\ast$.

**Theorem 1.1.2** For all $0 < |k| \leq S$, there is a feedback control

$$\psi_k = -\nu(L_k^{-2}R_kz_k)'(1)$$

such that, once inserted into the system (1.1.15), the corresponding solution of the closed loop system (1.1.15), satisfies

$$\|L_k^{-1}z_k(t)\| \leq Ce^{-\gamma t}\|L_k^{-1}z_0\|, \forall t \geq 0,$$

for some $C_\gamma, \gamma > 0$. Here $R_k \in L(X, X)$ is a linear self-adjoint operator such as

(i) $R_k : H \rightarrow H$,

(ii) $R_k$ satisfies the next algebraic Riccati type equation

$$\langle L_k^{-1}R_kz_0, L_k^{-1}A_kz_0 \rangle + \frac{1}{2}\nu^2|(L_k^{-2}R_kz_0)'(1)|^2 = \frac{1}{2}\|L_k^{-1}z_0\|^2, \forall z_0 \in H.$$

### 1.1.5 Feedback stabilization for the linearized system (1.1.2)

The main result of this section is

**Theorem 1.1.3** The feedback controller

$$\Psi(t, x) = -\nu \sum_{0 < |k| \leq S} (L_k^{-2}R_kL_kv_k(t))''(1)e^{ikx}, \quad (1.1.20)$$

where

$$v_k(t, y) = \int_0^{2\pi} v(t, x, y)e^{-ikx}dx, \quad 0 < |k| \leq S,$$

once inserted into equation (1.1.2), the corresponding solution to the closed-loop system (1.1.2) satisfies

$$\|v(t, v(t))\| \leq C_\alpha e^{-\alpha t}\|(u^0, v^0)\|^2, \quad t \geq 0,$$

for some $C_\alpha, \alpha > 0$.

### 1.2 Tangential feedback stabilization of periodic fluids in a two-dimensional channel

We consider again the Navier-Stokes equations in a two-dimensional channel, but now with the boundary control acting to the tangential component of the velocity field, on the upper wall.

$$\begin{cases}
    u_t - \nu \Delta u + uu_x + uv_y = p_x, \quad x \in \mathbb{R}, \ y \in (0, 1), \\
    v_t - \nu \Delta v + uv_x + vv_y = p_y, \quad x \in \mathbb{R}, \ y \in (0, 1), \\
    u_x + v_y = 0, \\
    u(t, x, 0) = 0, \ u(t, x, 1) = \Psi(t, x), \ v(t, x, 0) = v(t, x, 1) = 0, \\
    u(t, x, 2\pi, y) = u(t, x, y), \ v(t, x, 2\pi, y) = v(t, x, y), \\
    p(t, x, 2\pi, y) = p(t, x, y), \ \forall t \geq 0, \ \forall x \in \mathbb{R}, \ \forall y \in (0, 1).
\end{cases} \quad (1.2.1)$$
Again, the effort is to stabilize the Poiseuille profile, from the previous section. Thus, as before, we are lead to the study of the null stabilization of the system

\[
\begin{align*}
  u_t - \nu \Delta u + u_x U^e + u u_x + v u_y &= p_x, \quad x \in \mathbb{R}, y \in (0, 1), \\
  v_t - \nu \Delta v + v_x U^e + v v_x + v v_y &= p_y, \quad x \in \mathbb{R}, y \in (0, 1), \\
  u_x + v_y &= 0, \\
  u(t, x, 0) &= 0, \quad u(t, x, 1) = \Psi(t, x), \quad v(t, x, 0) = v(t, x, 1) = 0, \\
  u(t, x + 2\pi, y) &= u(t, x, y), \quad v(t, x + 2\pi, y) = v(t, x, y), \\
  p(t, x + 2\pi, y) &= p(t, x, y), \quad \forall t \geq 0, \forall x \in \mathbb{R}, \forall y \in (0, 1),
\end{align*}
\]

with its linearized given by

\[
\begin{align*}
  u_t - \nu \Delta u + u_x U^e + u u_x &= p_x, \\
  v_t - \nu \Delta v + v_x U^e &= p_y, \\
  u_x + v_y &= 0, \\
  u(t, x, 0) &= 0, \quad u(t, x, 1) = \Psi(t, x), \quad v(t, x, 0) = v(t, x, 1) = 0, \\
  u(t, x + 2\pi, y) &= u(t, x, y), \quad v(t, x + 2\pi, y) = v(t, x, y), \\
  p(t, x + 2\pi, y) &= p(t, x, y), \quad \forall t \geq 0, \forall x \in \mathbb{R}, \forall y \in (0, 1). 
\end{align*}
\]

Arguing as in the first section, we obtain the next stabilization result for the linearized system (1.2.3)

**Theorem 1.2.1** The feedback controller

\[
\Psi(t, x) = -\nu \sum_{0 < |k| \leq S} \frac{1}{i k} (L_k^{-2} R_k L_k v_k(t))''(1) e^{i k x},
\]

where

\[
v_k(t, y) = \int_0^{2\pi} v(t, x, y) e^{-i k x} dx, 0 < |k| \leq S
\]

once inserted into the system (1.2.3), the corresponding solution to the closed-loop system (1.2.3) satisfies the exponential decay

\[
\| (u(t), v(t)) \|^2 \leq C_\beta e^{-\beta t} \| (u^0, v^0) \|^2, \quad t \geq 0,
\]

for some $C_\beta, \beta > 0$.

**1.2.1 Local feedback stabilization for the full nonlinear Navier-Stokes system (1.2.2)**

Because of the tangential conditions, we are able in this case to prove also the local stabilization of the nonlinear system (1.2.2), using the fixed point method developed by Barbu and Triggiani [13].

**Theorem 1.2.2** Let $W := D(A^{\frac{1}{2}})$ and the neighbourhood of zero

\[
U_\rho := \{(u^0, v^0) \in W; \| (u^0, v^0) \| _W \leq \rho \}.
\]
The feedback controller

\[ \Psi(t, x) = -\nu \sum_{0 < |k| \leq S} \frac{1}{i k} (L_k^{-2} R_k L_k v_k(t))''(1)e^{ikx}, \]

once inserted into the system (1.2.2) implies the existence of a sufficiently small \( \rho > 0 \) such that for all initial data \((u^0, v^0) \in U_\rho\) there exists a unique solution

\[ (u, v) \in C([0, \infty); W) \cap L^2(0, \infty; Z), \]

of the closed-loop systema (1.2.2), which satisfies

\[ \| (u(t), v(t)) \|_W \leq Me^{-\omega t} \| (u^0, v^0) \|_W, \forall t > 0, \]

where \( Z := D(A^{\frac{3}{2}}) \). \( A \) is the Stokes operator

\[ A = -P \Delta, \ D(A) = H^2(Q) \cap H^1_0(Q) \cap H, \]

where \( P \) is the Leray projector.

1.3 Normal feedback stabilization of the periodic flows in a three-dimensional channel

The 2-D results presented above can be extended straightforward to the three-dimensional case. These results are obtained by the author in [67].
1.3.1 Presenting of the problem

The Navier-Stokes equations in a three-dimensional channel are given by

\[
\begin{align*}
&u_t - \nu \Delta u + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x}, \\
&v_t - \nu \Delta v + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -\frac{\partial p}{\partial y}, \\
&w_t - \nu \Delta w + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z}, \\
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,
\end{align*}
\]

for \( x, z \in \mathbb{R}, y \in (0, 1), t \geq 0 \).

\[(1.3.1)\]

Again the effort is to stabilize the parabolic Poiseuille profile

\[
U^e = -\frac{a}{2\nu}(y^2 - y), \quad V^e = 0, \quad W^e = 0, \quad y \in [0, 1].
\]

\[(1.3.2)\]

Decomposing the linearized system of (1.3.1), around the steady-state solution, in Fourier modes, reducing the pressure and using the divergence free condition we obtain

\[
\begin{align*}
&v''_{tkl} - (k^2 + l^2)v_{tkl} - \nu v'''_{tkl} + [2\nu(k^2 + l^2) + ikU^e]v''_{tkl} \\
&\quad - \nu(k^2 + l^2)^2 + ik(k^2 + l^2)U^e + ik(U^e)'v_{tkl} = 0, \\
&t \geq 0, \quad y \in (0, 1), \\
&v'_{kl}(0) = v'_{kl}(1) = 0, \quad v_{kl}(0) = 0, \quad v_{kl}(1) = \psi_{kl}(t).
\end{align*}
\]

\[(1.3.3)\]

We immediately notice the similarities between this system and (1.1.4). Therefore, arguing as before and introducing the operators \( L_{kl}, F_{kl} : \mathcal{D}(L_{kl}) \subset H \to H \) and \( A_{kl} : \mathcal{D}(A_{kl}) \subset H \to H \), defined as

\[
\begin{align*}
&L_{kl}v := -v'' + (k^2 + l^2)v, \quad \mathcal{D}(L_{kl}) = H^1_0(0, 1) \cap H^2(0, 1), \\
&F_{kl}v := \nu v''' - (2\nu(k^2 + l^2) + ikU^e)v'' + (\nu(k^2 + l^2)^2 + ik(k^2 + l^2)U^e + ik(U^e)')v,
\end{align*}
\]

\[(1.3.4) \quad (1.3.5)\]
\[ D(F_{kl}) = H^4(0,1) \cap H^2_0(0,1), \]
\[ A_{kl} := F_{kl}L_{kl}^{-1}, \quad D(A_{kl}) = \{ v \in H : L_{kl}^{-1}v \in D(F_{kl}) \}, \]

respectively, we get

**Theorem 1.3.1** The feedback controller

\[ \Psi(t, x, z) = \sum_{k,l \in \mathbb{Z}} \psi_{kl}(t) e^{ikx}e^{ilz}, \quad (1.3.7) \]

where

\[ \psi_{kl}(t) = \begin{cases} 
-\nu(L_{k}^{-2}R_kL_kv_{k0}(t))''(1) & \text{for } 0 < |k| \leq S, \ l = 0, \\
0 & \text{for } |k| > S, \ l = 0, \\
0 & \text{for } k = 0, \ l \in \mathbb{Z}, \\
-\nu(L_{kl}^{-2}R_{kl}L_{kl}v_{kl}(t))''(1) & \text{for } k, l \in \mathbb{Z}^* \text{ si } \sqrt{k^2 + l^2} \leq S, \\
0 & \text{for } k, l \in \mathbb{Z}^* \text{ si } \sqrt{k^2 + l^2} > S, 
\end{cases} \quad (1.3.8) \]

once inserted into the linearized system of (1.3.1), yields the exponential decay for the corresponding solution

\[ \|(u(t), v(t), w(t))\|^2 \leq C_\gamma e^{-\gamma t}\|(u^0, v^0, w^0)\|^2, \quad t \geq 0, \]

for some \( C_\gamma, \gamma > 0. \)

### 1.4 Normal feedback stabilization of a MHD flow in a two-dimensional and three-dimensional channel

In this last section, we treat the case when the fluid is electrically conductive and an external magnetic field is applied transversal to the channel. These results are obtained by the author in [70].

**1.4.1 Main results**

The governing equations of the dynamics are the MHD equations, a combination between the Navier-Stokes equations and Maxwell equations, i.e.,

\[
\frac{\partial \mathbf{v}}{\partial t} - \nu \Delta \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} - \mathcal{N}(\mathbf{j} \times \mathbf{B}) = -\nabla p, \quad (1.4.1)
\]

and

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{1}{Rm} \Delta \mathbf{B}, \quad (1.4.2)
\]

where \( \mathbf{v} \) is the velocity field, \( \mathbf{B} \) is the magnetic field, \( \mathbf{j} \) is the current density and \( p \) is the pressure \( \nu, Rm \) and \( \mathcal{N} \) are the viscosity coefficient, the magnetic Reynolds number and the Stuart number, respectively. We shall consider the fluid with low values of the magnetic
Reynolds number. It turns out that the MHD equations (1.4.1)-(1.4.2) transform into the so-called simplified MDH equations, given by

\[
\begin{align*}
&\begin{cases}
  u_t - \nu \Delta u + \mathcal{N}u + uu_x + vu_y = -p_x, \\
v_t - \nu \Delta v + uv_x + vv_y = -p_y, \\
u_x + v_y = 0,
\end{cases} \\
&\quad t \geq 0, x \in \mathbb{R}, y \in (0, 1),
\end{align*}
\]

(1.4.3)
in 2-D,

\[
\begin{align*}
&\begin{cases}
  U_t - \nu \Delta U + UU_x + VU_y + WW_z - \mathcal{N}\phi_z + \mathcal{N}U = -P_x, \\
V_t - \nu \Delta V + UV_x + VV_y + WV_z = -P_y, \\
W_t - \nu \Delta W + UW_x + VW_y + WW_z + \mathcal{N}\phi_x + \mathcal{N}W = -P_z, \\
\Delta \phi = W_x - U_z, \\
U_x + V_y + W_z = 0,
\end{cases} \\
&\quad t \geq 0, x, z \in \mathbb{R}, y \in (0, 1),
\end{align*}
\]

(1.4.4)
in 3-D, respectively.

The steady-state solutions which are stabilized are the Hartmann-Poiseuille profiles, given by

\[
U^e = \frac{\sinh \left( \sqrt{\frac{1}{\nu} \mathcal{N}} (1-y) \right) - \sinh \sqrt{\frac{1}{\nu} \mathcal{N}} + \sinh \left( \sqrt{\frac{1}{\nu} \mathcal{N}} y \right)}{2 \sinh \left( \sqrt{\frac{1}{\nu} \mathcal{N}} / 2 \right) - \sinh \sqrt{\frac{1}{\nu} \mathcal{N}}}, \quad V^e \equiv 0,
\]

(1.4.5)
in 2-D,

\[
U^e = \frac{\sinh \left( \sqrt{\frac{1}{\nu} \mathcal{N}} (1-y) \right) - \sinh \sqrt{\frac{1}{\nu} \mathcal{N}} + \sinh \left( \sqrt{\frac{1}{\nu} \mathcal{N}} y \right)}{2 \sinh \left( \sqrt{\frac{1}{\nu} \mathcal{N}} / 2 \right) - \sinh \sqrt{\frac{1}{\nu} \mathcal{N}}}, \quad V^e \equiv 0, \quad W^e \equiv 0,
\]

(1.4.6)
in 3-D, respectively.

Let us notice the clear similarities between the SMHD equations and the ones studied in the previous sections. One might suspect that, arguing as before, one can obtain a normal feedback stabilization result as before. This is indeed, making use of the Fourier functional setting, one can argue as before in order to obtain the wanted results.
Chapter 2

Internal stabilization of Navier-Stokes equations with exact controllability on spaces with finite co-dimensiona

In this chapter, we design an internal stabilizing feedback controller for the Navier-Stokes equations. Besides, we show that this controller steers the initial data into the space of stable modes, in finite time. These results were obtained by the author in [21], jointly with V. Barbu.

2.1 Presentation of the problem

The Navier-Stokes equations with null boundary conditions are

\[
\begin{align*}
\frac{\partial \mathbf{v}}{\partial t} - \nu \Delta \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} &= f^e + \nabla p \text{ in } (0, \infty) \times \mathcal{O}, \\
\nabla \cdot \mathbf{v} &= 0 \text{ in } (0, \infty) \times \mathcal{O}, \\
\mathbf{v} &= 0 \text{ on } (0, \infty) \times \partial \mathcal{O}, \\
\mathbf{v}(0) &= \mathbf{v}_0 \text{ in } \mathcal{O},
\end{align*}
\]

where \( \mathcal{O} \subset \mathbb{R}^d, \ d = 2, 3, \) is an open domain with smooth boundary \( \partial \mathcal{O}. \)

We consider a steady-state solution \( \mathbf{v}^e = \mathbf{v}^e(x) \) of N-S equations, i.e.,

\[
\begin{align*}
-\nu \Delta \mathbf{v}^e + (\mathbf{v}^e \cdot \nabla) \mathbf{v}^e &= f^e + \nabla p^e \text{ in } \mathcal{O}, \\
\nabla \cdot \mathbf{v}^e &= 0 \text{ in } \mathcal{O}; \ \mathbf{v}^e = 0 \text{ pe } \partial \mathcal{O}.
\end{align*}
\]
Let \( O_0 \subset O \), we associate to the system (2.1.1) the next internal control problem

\[
\begin{aligned}
\frac{\partial v}{\partial t} - \nu \Delta v + (v \cdot \nabla) v &= f^e + \nabla p + m \Psi \quad \text{in} \ (0, \infty) \times O, \\
\nabla \cdot v &= 0 \quad \text{in} \ (0, \infty) \times O, \\
v &= 0 \quad \text{on} \ (0, \infty) \times \partial O, \\
v(0) &= v_o \quad \text{in} \ O,
\end{aligned}
\tag{2.1.3}
\]

where \( m \) is the characteristic function of the set \( O_0 \) and \( \Psi \) is the control.

### 2.2 Main results

We set

\[
H_\pi := \left\{ v \in (L^2(O))^d : \nabla \cdot v = 0 \text{ in } O, \ v \cdot n = 0 \text{ on } \partial O \right\},
\tag{2.2.1}
\]

where \( n \) is the unit outward normal to the boundary \( \partial O \). The Leray projector is defined as \( P : (L^2(O))^d \to H_\pi \). We introduce also

\[
A := -P\Delta, \quad D(A) = (H^1_0(O) \cap H^2(O))^d \cap H_\pi,
\]

\[
A \v := \nu A \v + P((\v \cdot \nabla)\v^e + (\v^e \cdot \nabla)\v), \quad D(A) = D(A),
\]

and

\[
B \v := P((\v \cdot \nabla)\v), \quad \v \in D(A).
\]

Thus, redefining \( \v := P(v - v^e) \), we rewrite the system (2.1.3) as

\[
\begin{aligned}
\frac{d}{dt} \v + A \v + B \v &= P(m \Psi) \quad \text{in} \ (0, \infty) \times O, \\
v(0) &= v_o := v_o - v^e,
\end{aligned}
\tag{2.2.2}
\]

since by applying the Leray projector we reduce the pressure.

Next, we introduce the feedback controller

\[
\Psi(t) := -\eta \sum_{j=1}^N \text{sign} \left( \langle P_N \v(t), \phi_j^* \rangle \right) P_N \Phi_j,
\tag{2.2.3}
\]

where \( \eta \in \mathbb{R}_+ \), \( \text{sign} \) is the multivalued function on \( \mathbb{C} \), defined as

\[
\text{sign}(z) := \begin{cases} \frac{z}{|z|}, & \text{if } z \neq 0, \\ \{ w \in \mathbb{C} : |w| \leq 1 \}, & \text{if } z = 0, \end{cases}
\tag{2.2.4}
\]

and \( \Phi_j \in H_\pi \) are defined as

\[
\Phi_j := \sum_{k=1}^N \alpha_{jk} \phi_k^*, \quad j = 1, \ldots, N,
\]

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with
\[\sum_{k=1}^{N} \alpha_{ik} \langle \phi_k^*, \phi_j^* \rangle_0 = \delta_{ij}, \ i, j = 1, \ldots, N. \tag{2.2.5}\]

Here
\[\langle \phi, \psi \rangle_0 := \int_{\mathcal{O}} m\phi \psi d\xi, \forall \phi, \psi \in (L^2(\mathcal{O}))^d.\]

We have the next result of stabilization for the N-S system

**Theorem 2.2.1** Let \(T, \rho > 0\) sufficiently small. For all \(v_o \in W\), such that \(\|v_o\|_W \leq \rho\), the problem

\[
\begin{cases}
\frac{dv}{dt} + \mathbf{A}v + \eta \sum_{j=1}^{N} \text{sign}(\langle P_Nv, \phi_j^* \rangle)P_Nm(\Phi_j) + Bv = 0, \ t \geq 0, \\ v(0) = v_o,
\end{cases}
\tag{2.2.6}
\]

is well-posed on \(W\) with unique solution
\[v \in C([0, \infty); W) \cap L^2(0, \infty; Z),\]

provided that \(\eta\) is such that
\[\eta \geq \max \left\{ \frac{\Re \lambda_j (k\|\phi_j^*\| + \rho)}{e^{\Re \lambda_j T} - 1}; j = 1, \ldots, N \right\}. \tag{2.2.7}\]

Besides, these solutions satisfy
\[P_Nv(t) = 0, \forall t \geq T, \tag{2.2.8}\]

and
\[\|v(t)\| \leq C e^{-\beta t}\|v_o\|, \forall t \geq T. \tag{2.2.9}\]

Here
\[W := (H^{\frac{1}{2}+\epsilon}(\mathcal{O}))^d \cap H_\pi \text{ and } Z := (H^{\frac{1}{2}-\epsilon}(\mathcal{O}))^d \cap H_\pi\]

for \(d = 2,\)
\[W := (H^{\frac{1}{2}+\epsilon}(\mathcal{O}))^d \cap H_\pi \text{ and } Z := (H^{\frac{1}{2}+\epsilon}(\mathcal{O}))^d \cap H_\pi\]

for \(d = 3.\)
Chapter 3

Internal stabilization of a finite set of steady-state solutions to the Na-vier-Stokes equations

In this chapter, we shall design a control which stabilizes, not one steady-state solution, but a finite set of steady-state solutions to the N-S system. The results presented below were obtained by the author in [69].

3.1 Main results

Remember the controlled N-S equations with zero boundary conditions

\[
\begin{aligned}
\begin{cases}
\dot{v}(x,t) - \nu \Delta v(x,t) + (v \cdot \nabla)v(x,t) \\
\quad = m(x)\Psi(x,t) + f^e(x) + \nabla p(x,t), \quad \text{in } Q = \mathcal{O} \times (0,\infty), \\
\nabla \cdot v = 0, \quad \text{in } Q, \\
v = 0, \quad \text{on } \Sigma = \partial \mathcal{O} \times (0,\infty), \\
v(x,0) = v_0, \quad \text{in } \mathcal{O}.
\end{cases}
\end{aligned}
\]

(3.1.1)

Applying the Leray projector, these can be rewritten as

\[
\frac{dv}{dt} + \nu Av + Bv = P(m\Psi_i(v - v^e_i)) + Pf^e, \quad t \geq 0; \quad v(0) = v_0 \in H_\pi.
\]

(3.1.2)

The main result of internal stabilization in [10], says that there exists a neighbourhood \(U_\rho\) of zero and a feedback controller which stabilizes system (3.1.2) in it. Moreover, we know that N-S obeys the next generic property: for "almost all" external forces \(f_e\), the number of steady-state solutions is finite. Thus, let \(\{v^e_1, v^e_2, \ldots, v^e_N\}\) the steady-state solutions. For each \(v^e_i, i = 1, \ldots, N\), there is a feedback controller \(\Psi_i = \Psi_i(v - v^e_i), i = 1, \ldots, N\), such that the solution to

\[
\frac{dv}{dt} + \nu Av + Bv = P(m\Psi_i(v - v^e_i)) + Pf^e, \quad t \geq 0; \quad v(0) = v_0,
\]

(3.1.3)
satisfies

$$|v(t) - v_i^e|_{\frac{1}{2}} \leq C_i e^{-\gamma t} |v_0 - v_i^e|_{\frac{1}{2}}, t \geq 0,$$

(3.1.4)

for $v_0 \in U_{\rho_i}$. Here we denote by $|v|_{\frac{1}{2}} := \|A^{\frac{1}{4}} v\|, \forall v \in \mathcal{D}(A^{\frac{1}{4}})$.

Consider the sets

$$U_i = \left\{ v_0 \in H_\pi; |v_0 - v_i^e|_{\frac{1}{2}} < \frac{\rho_i}{C_i} \right\}, i = 1, ..., N,$$

(3.1.5)

and $\epsilon > 0$ such that

$$\left\{ v; |v - v_i^e|_{\frac{1}{2}} < (1 + \epsilon)\rho_i \right\} \cap \left\{ v; |v - v_j^e|_{\frac{1}{2}} < (1 + \epsilon)\rho_j \right\} = \emptyset,$$

(3.1.6)

$\forall j \neq i, i, j = 1, ..., N$.

Introduce the function $w : \mathbb{R}_+ \to [0, 1]$, defined as

$$w(r) = \begin{cases} 
1 & 0 \leq r \leq 1, \\
0 & r \geq 1 + \epsilon, \\
\text{netedá} & 1 < r < 1 + \epsilon.
\end{cases}$$

Finally, introduce $\chi_i : \mathcal{D}(A^{\frac{1}{4}}) \to [0, 1]$, as

$$\chi_i(v) = w \left( \frac{|v|_{\frac{1}{2}}}{\rho_i} \right), \forall v \in \mathcal{D}(A^{\frac{1}{4}}), i = 1, ..., N.$$  

(3.1.7)

Then, the controller

$$\Psi(v) := \sum_{i=1}^{N} \chi_i(v - v_i^e) \Psi_i(v).$$

(3.1.8)

stabilizes the finite-set of steady-state solutions $v_i^e$.

Theorem 3.1.1 The feedback controller $\Phi$, defined by (3.1.8), exponentially stabilizes every stationary solution $v_i^e, i = 1, ..., N$ in the neighbourhood $U_i, i = 1, ..., N$. More precisely, for all $v_0 \in U_i, i = 1, ..., N$ there exists a unique weak solution

$$v \in L^\infty(0, T; H_\pi) \cap L^2(0, T; V), \frac{dv}{dt} \in L^4(0, T; V^*),$$

for $d = 3$, and

$$v \in L^\infty(0, T; H_\pi) \cap L^2(0, T; V), \frac{dv}{dt} \in L^2(0, T; V^*)$$

for $d = 2, \forall T > 0$, to the closed loop system

$$\frac{dv}{dt} + \nu Av + Bv = P(m \sum_{j=1}^{N} \chi_j(v - v_j^e) \Psi_j(v)) + Pf^e, \ t \geq 0; v(0) = v_0,$$

(3.1.9)

which satisfies the exponential decay

$$|v(t) - v_i^e|_{\frac{1}{2}} \leq C_i e^{-\gamma t} |v_0 - v_i^e|_{\frac{1}{2}}, t \geq 0.$$
Chapter 4

Appendix

This chapter briefly presents the main theoretical results used during the presentation. More precisely, we remember results about compact operators in Banach spaces, spectral decomposition, Fredholm theory, the extended of an operator, semigroup theory and dynamical systems theory.
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